Adolescent Reasoning in Mathematics: Exploring Middle School Students' Strategic Approaches in Empirical Justifications

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Background

- In mathematics, formal reasoning is deductive. However, middle school students struggle with this approach.
- Their tendency is to reason empirically, using examples to show the truth of a mathematical conjecture (e.g., Healy & Hoyles, 2000; Knuth et al., 2009). Ultimately, the goal is to consider how understanding inductive reasoning can support learning deductive approaches.
- In other domains, inductive reasoning is a valuable and reasonable strategy, with its success depending on knowledge about category structure (Osherson et al., 1990).
- Here, we ask how children apply intuitions and strategies within their empirical justifications in mathematics. The overall goal is to consider how students' strategic use of inductive reasoning and deductive reasoning interact.

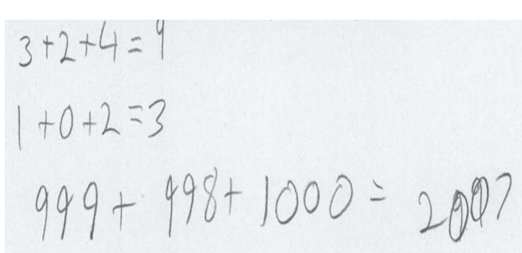
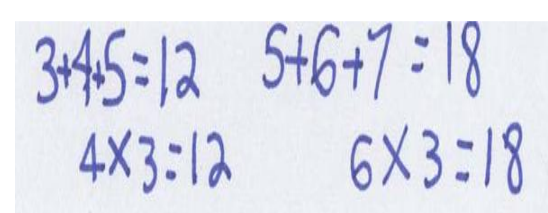
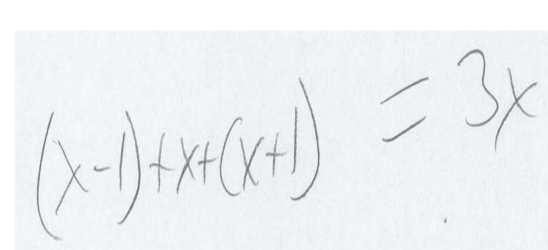
Method

- Semi-structured interviews with 20 middle school students (11 F, 9 M)
- Two mathematical conjectures:

Whole Number Conjecture:
First, pick any whole number.
Second, add this number to the number before it and the number after it.
Your answer will always equal 3 times the number you started out with.

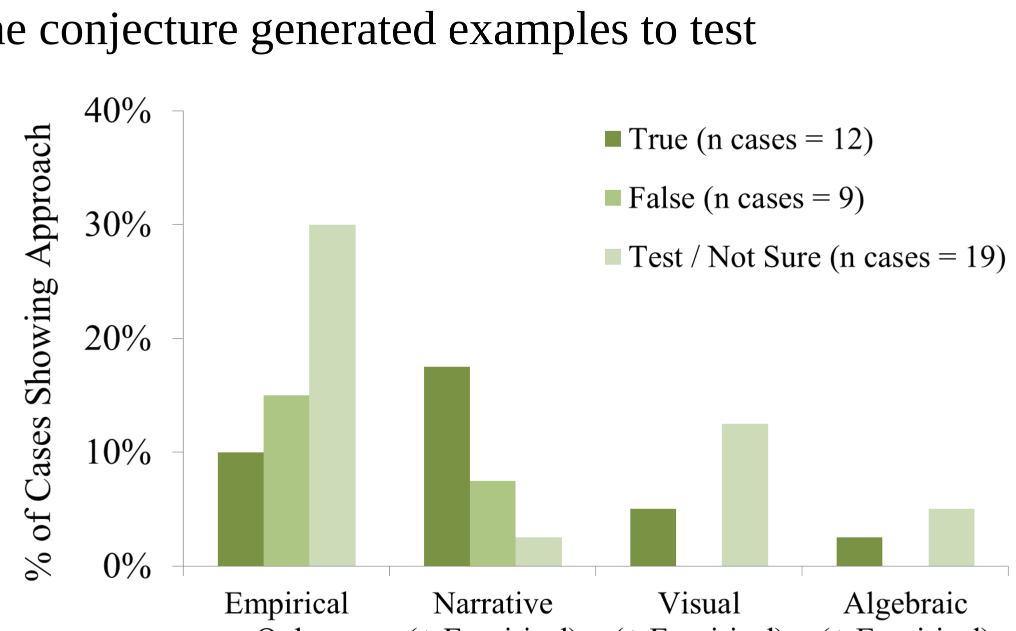
Even Number Conjecture:
First, pick any even number.
Second, add this number to half of itself.
Your answer will always be divisible by 3.

Types of Approaches to Figuring Out A Conjecture's Truth

Empirical	 
Narrative	"Because you're adding- you're adding the number to the number before it and then to the number after it, which is like, the number after it is one more and the number before it is one less. So if you add one and take away one, you're not really doing anything to it."
Visual	"The one in the middle has four tallies. The one on the right of it has five. But- the one on the right of it has five, but the one on the left of it has three. So if you take the one from this- from the five and put it onto the three. This one's four. Well this one's four. And then they all equal four. So that's three times." 
Algebraic	"The- our whole number equals X. And so basically what it's sayin' is- the number before it basically equals X minus one. {Okay} If you think about that. Plus, you know, X. And then to- you add the number that comes after it would be one plus you're first number. So you get X plus one. Now to simplify this… we can agree on that 3X is equal to X plus X plus X." 

How did students approach the conjectures?

- 47% unwilling to specify initial belief (increased with age)
- All but one student on one conjecture generated examples to test
- 18 (out of 40 possible) cases of valid generalizable proofs
- Valid proofs more likely with initial reactions of true



Were Valid Generalizable Proofs a Good Approach?

If available, a proof was the preferred method for showing someone else.

"…more convincing 'cause it's a little bit more in general. And it's not using like one specific number. It's giving a rule kind of. [It's] using a variable to some extent."

"It's way more convincing than all that stuff [trying examples]. Now that I can like see how it works out instead of just like finding, oh, it does work out."

Features of Students' Tested Example Sets

- More than one example needed to be tested

"Because I think I tried it enough times to prove it."

"So if you tried it with a thousand numbers, you're gonna have better data than if you just tried it with three"

- Diverse example sets: Variation expressed in tested examples and justifications

	Varied Magnitude	Varied Parity
Whole number conjecture	37%	74%
Even number conjecture	75%	-

"it would be good to try out even and odd and different kinds of numbers."

"so people might say like if you picked all even numbers people might say what about odd numbers or something like that"

- Using both typical and unusual numbers was deemed a good approach
  - 'typical' numbers used by all students
  - 'unusual' numbers used by 75% of students

"I wanted to do numbers that were hard for it. You know. Like numbers that you can't really- that are prime…"

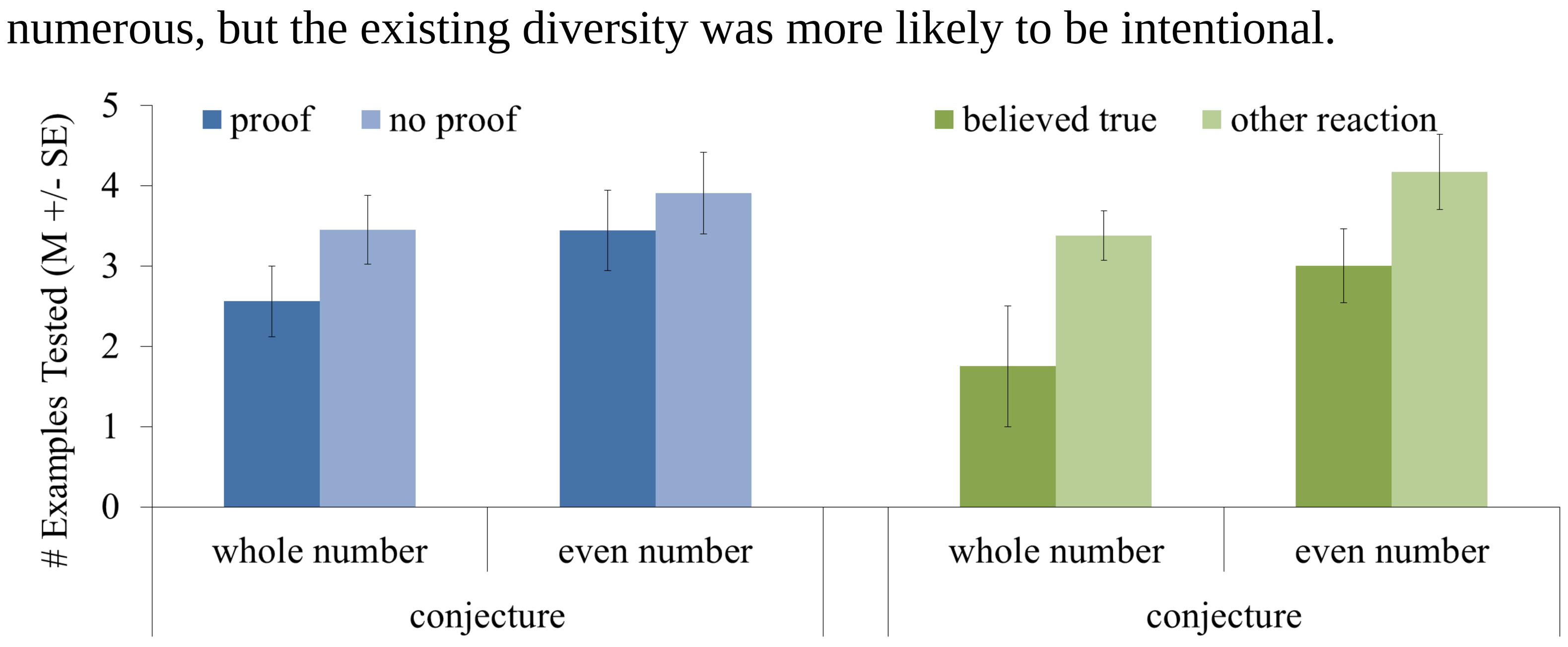
- odd, prime, large, or uncommon numbers were considered unusual by students

- Majority of the tested example sets had both mathematically special and ordinary numbers

Students expressed the value of varying these dimensions by impliciting choosing sets meeting these characteristics, justifying their choices, and evaluating their own approaches: increasing the quantity and/or diversity even further was the preferred method for showing someone else if using examples.

How were examples used strategically?

With true beliefs or with valid proofs, example sets were less diverse and less numerous, but the existing diversity was more likely to be intentional.



Conclusions

In summary, middle school students reasoning about mathematics conjectures

- are influenced by their initial reaction to a conjecture's validity.
- understand that *principled variation* of examples offers stronger evidence.
- *strategically* combine empirical and more generalizable approaches.
- recognize that deductive proof, when available to them, is more convincing.

As in other domains, the example sets and justifications students offered showed a strategic approach to inductive reasoning based on knowledge about mathematical properties of numbers.

Building on students' strengths in implementing inductive reasoning could help support a transition to deductive reasoning.