

How Students Reason Differently in Everyday and Mathematical Contexts: Typicality and Example Choice in Justification

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The Case of 1729

The mathematician G.H. Hardy was visiting his protégé, the Indian mathematician Ramanujan in the hospital. To make small talk, he remarked that 1729, the number of the taxi which had brought him, was rather dull number.

Ramanujan replied immediately *“No Hardy! No Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways.”*

$$(1729 = 1^3 + 12^3 = 9^3 + 10^3)$$

“Every positive integer is one of Ramanujan's personal friends.”

J. E. Littlewood



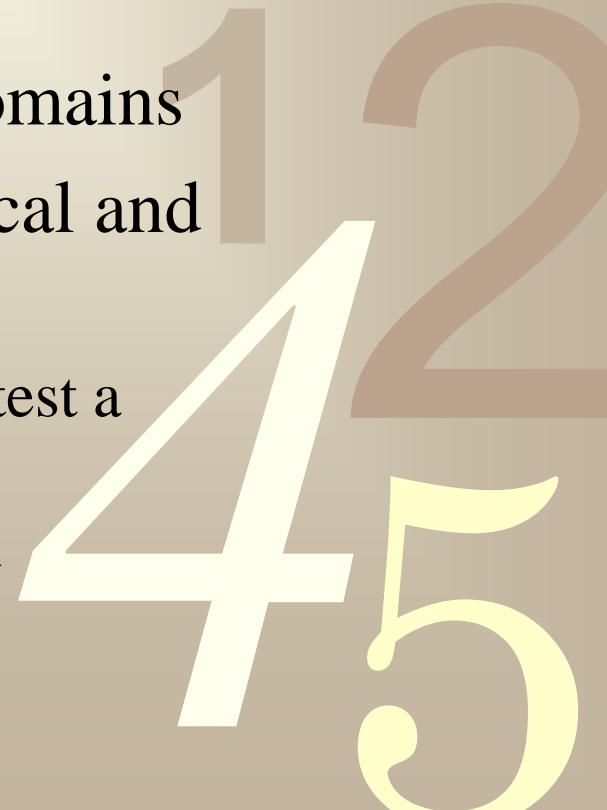
The Case of 1729

- Mathematical objects, like numbers, hold different meanings for different people
 - Considerations mathematically significant
 - May be consistently related to expertise
- Interact with and use these mathematical objects as we explore conjectures
- **How do novices and experts think about mathematical objects, like numbers, differently when considering conjectures?**

Overview

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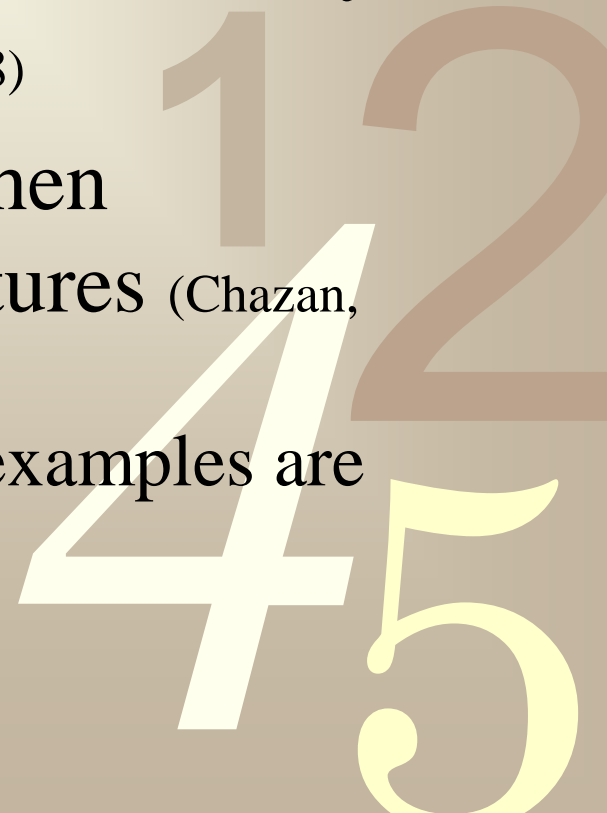
- Mathematical justification/proof
- Using examples to explore mathematical conjectures
- Example-based reasoning in other domains
- *Typicality* of examples in mathematical and non-mathematical domains
 - Using typical and atypical examples to test a conjecture
- Research Questions & Current Study



Mathematical Justification/Proof

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- *“The process employed by an individual to remove or create doubts about the truth of an observation”* (Harel & Sowder, 1998)
- Students often test examples when exploring mathematical conjectures (Chazan, 1993; Healy & Hoyles, 2000; Knuth et al., 2011)
 - Some students may believe that examples are sufficient as mathematical proof



“The sum of any three consecutive whole numbers is three times the middle number.”

Student A:

$$3+4+5=12$$
$$4 \times 3 = 12$$



Student B:

$$3+2+4=9$$

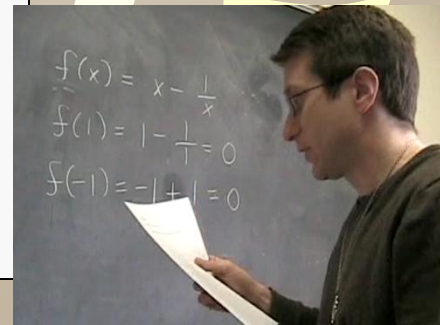
$$1+0+2=3$$

$$999 + 998 + 1000 = 2997$$

Mathematical Justification

- Mathematicians use examples when exploring conjectures too (Alcock, 2004)
- Verify and understand conjectures, generalize from examples to proof, seek counter-examples to “break” the conjecture (Lockwood et al., 2012)

“I start with an easy, typical example, to test the conjecture in that case. Second, I will work with a more complex but still fairly typical example. If these examples seem to be in accordance with the conjecture, then I will next try to test some strange or pathological examples, to really push the boundaries of what might be possible in this situation... often, counterexamples to a conjecture will be found among the strangest things that can happen in a situation.”



Mathematical Justification

- Example-based reasoning important in other domains (Osherson et al., 1990; Feeny & Heit, 2007)
- Conjecture: **Birds have hollow bones**
- What does “good” example-based reasoning look like?
 - *Quantity* - More examples better than fewer



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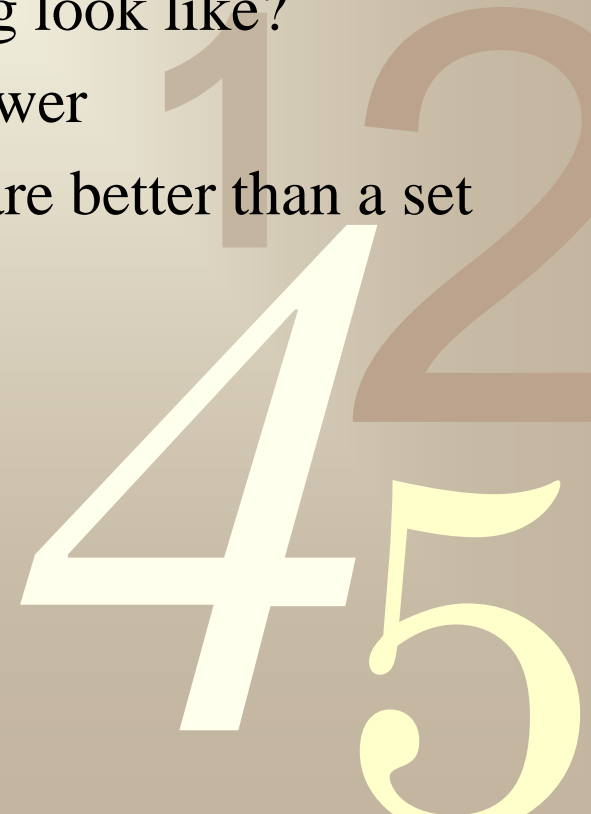
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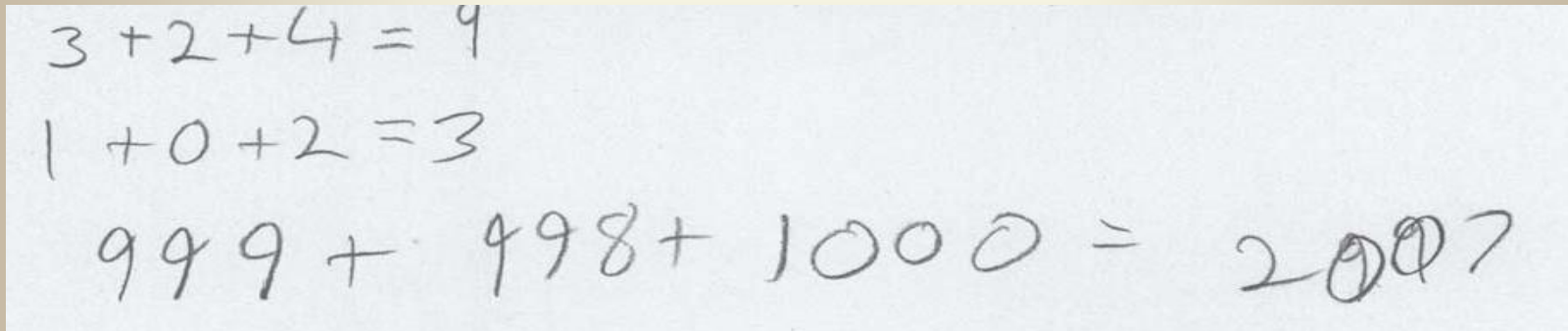
Mathematical Justification

- Example-based reasoning important in other domains (Osherson et al., 1990; Feeny & Heit, 2007)
- Conjecture: **Birds have hollow bones**
- What does “good” example-based reasoning look like?
 - *Quantity* - More examples better than fewer
 - *Diversity* - A wide variety of examples are better than a set of very similar examples
 - *Typicality* - Generic or ‘average’ examples generalize better than special or ‘weird’ examples



Typicality

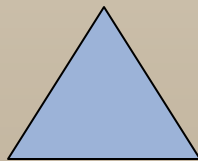
- Middle school students report choosing **more typical** and **less typical** examples when given conjectures (Cooper et al., 2011)



Handwritten mathematical examples on a white background:

$$3 + 2 + 4 = 9$$
$$1 + 0 + 2 = 3$$
$$999 + 998 + 1000 = 2007$$

- Unclear if realize that properties that hold for mathematically “typical” objects more likely to generalize to other objects
 - i.e., Testing a conjecture about triangles using only an equilateral



Typicality

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- What considerations make an object “typical” or “atypical” to middle school students is also unclear.

“Ten is like so easy. And things are sold in 10's a lot. And 10's like an easy number to work with and add 'cause it ends with 0.”

“Seven is] odd. It's not counted by frequently. It's not used very frequently. Like I'm in dance. So I think of things in beats a lot.”

- Mathematical properties?
- Everyday familiarity?



Typicality

- *Everyday typicality*: How common an object is in an individual's everyday life



- *Mathematical typicality*: How typical an object is when its mathematical properties are considered in relation to the properties of *all* objects of that type

$$0 \times n = 0$$

$$1 \times n = n$$

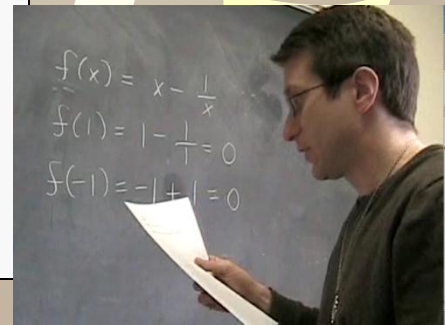
Typicality

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- Mathematicians also report varying typicality (Lockwood et al., this volume)
 - Common examples with no special properties
 - Unusual, obscure, or “tricky” examples
 - Examples that are special or boundary cases

“I start with an easy, typical example, to test the conjecture in that case. Second, I will work with a more complex but still fairly typical example. If these examples seem to be in accordance with the conjecture, then I will next try to test some strange or pathological examples, to really push the boundaries of what might be possible in this situation... often, counterexamples to a conjecture will be found among the strangest things that can happen in a situation.”

1
2



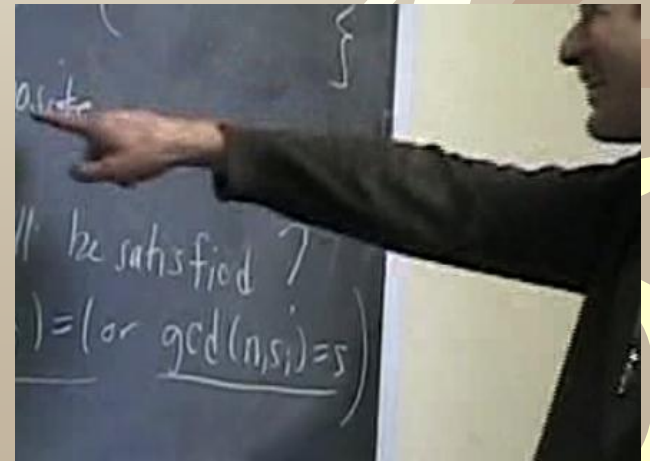
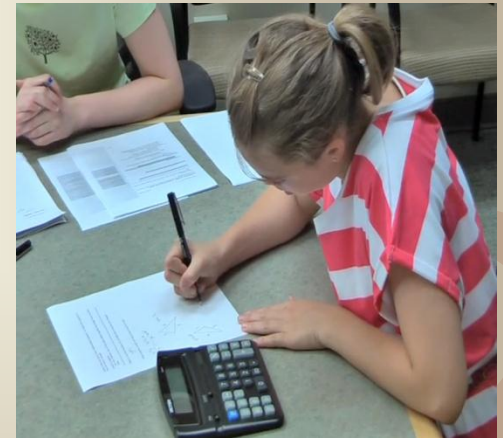
Research Questions

- 1) Do middle school students and mathematicians have distinct notions of everyday and mathematical typicality?
- 2) Do middle school students and mathematicians use typicality *strategically* when considering mathematical conjectures?
 - Special mathematical properties should decrease mathematical typicality
 - More superficial or everyday properties should not impact mathematical typicality

Participants

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- 475 middle school students from a suburban Midwestern school (grades 6-8)
 - Paper survey
- 339 mathematicians recruited via email from university mathematics departments
 - Online survey



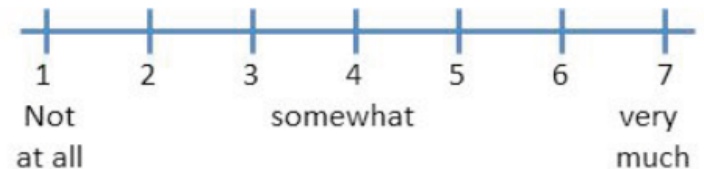
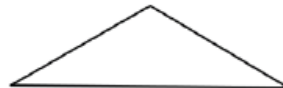
Survey Instrument

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- Rate typicality of **numbers, triangles, and parallelograms** on 1-7 scale
- 2 Contexts: **Everyday** and **Mathematical**

Now we want you to think about triangular shapes you see in your everyday life outside of school, for example, in objects around your house or outside.

1. How typical is this triangle of those you see in your everyday life?



Survey Instrument

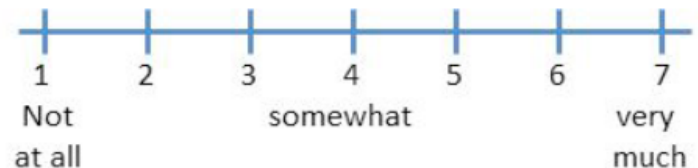
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- Rate typicality of **numbers, triangles, and parallelograms** on 1-7 scale
- 2 Contexts: Everyday and **Mathematical**

Now we want you to think about mathematical properties - that is, the kinds of things you learn about in math class.

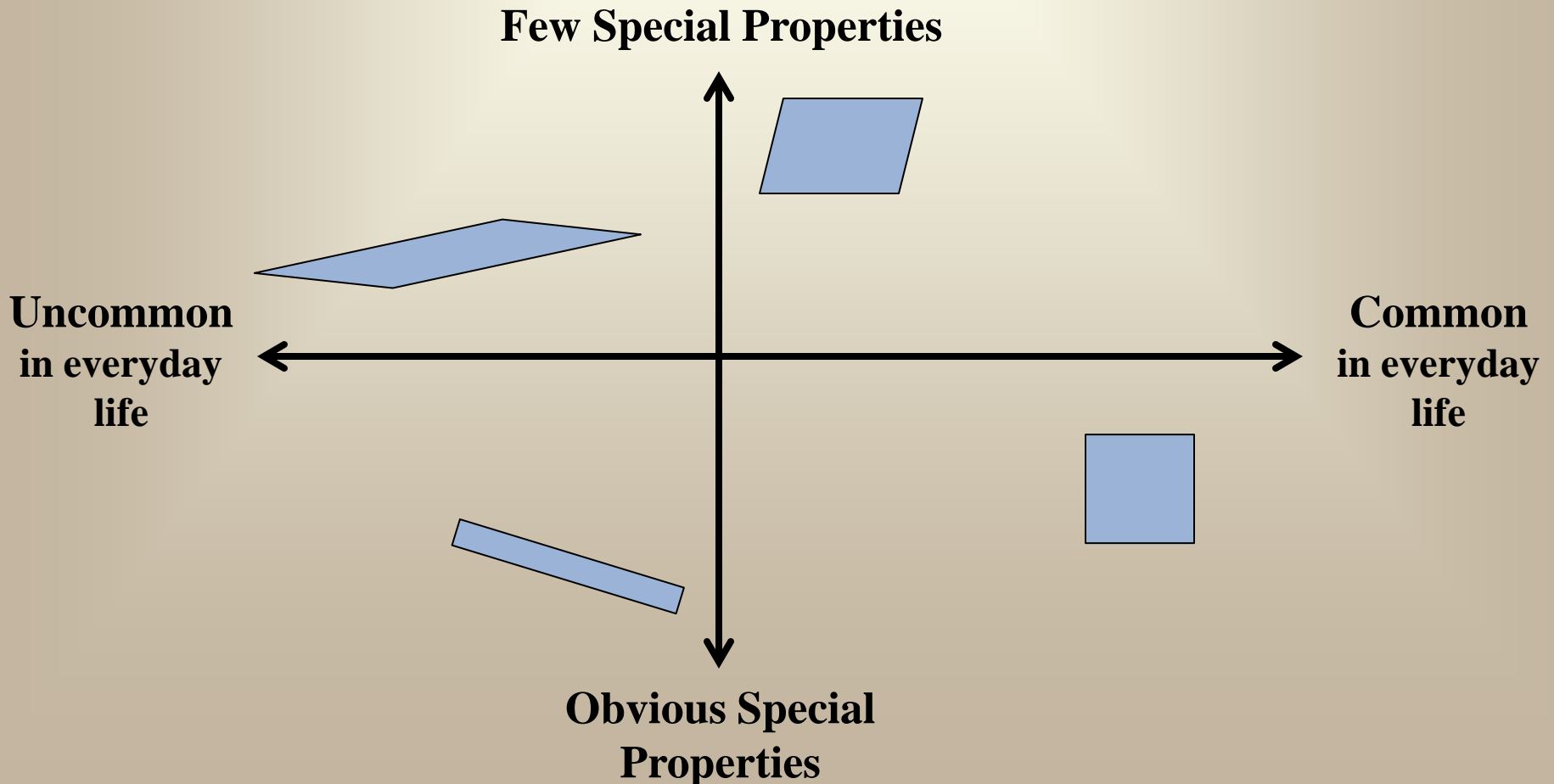
1. Imagine that we learned a new mathematical property that was true of this number. How likely is it that the property will be true of most other numbers?

9



Survey Instrument

- 22 parallelograms, 22 triangles, 27 numbers

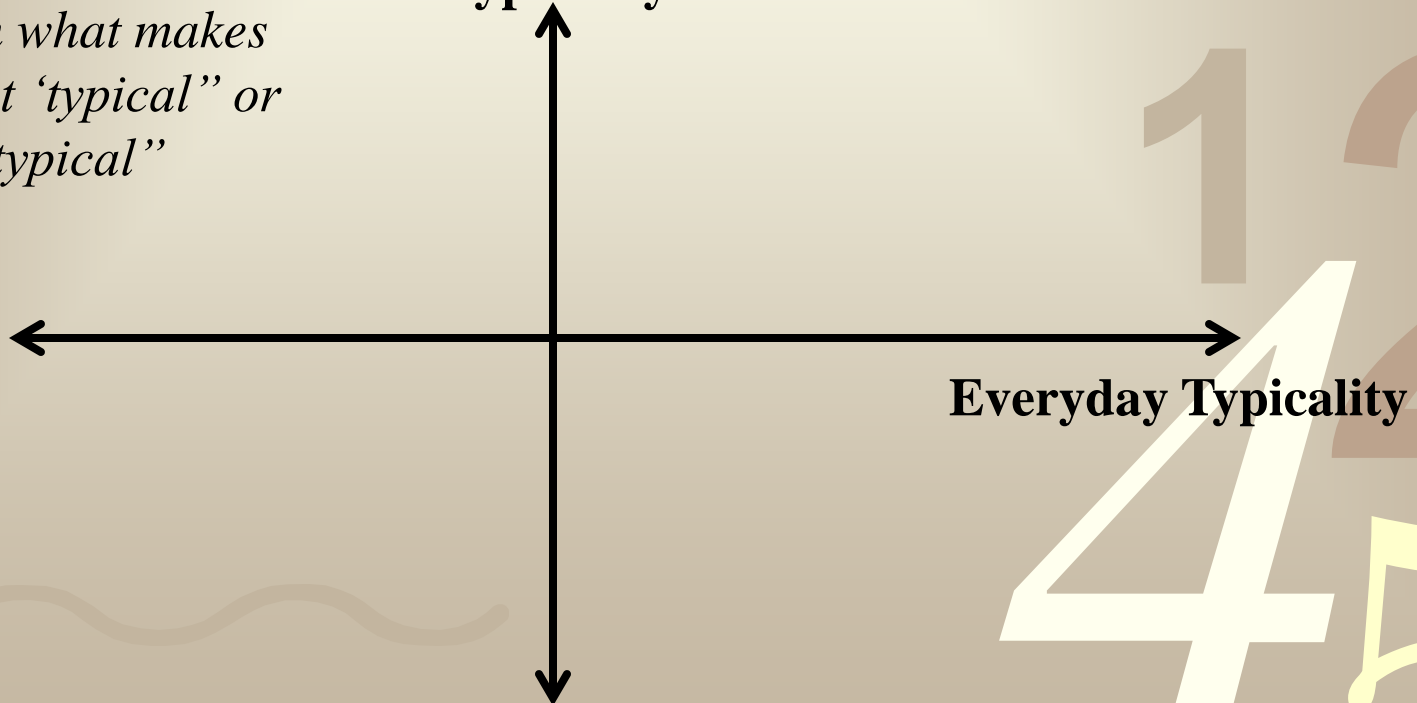


Survey Instrument

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Based heavily on results from previous work on what makes an object 'typical' or 'atypical'

**Mathematical
Typicality**



Everyday Typicality

Survey Instrument

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	Numbers	Parallelograms	Triangles
Mathematical Properties	<ul style="list-style-type: none">• Prime• Perfect square• Power of 2 or 10• Multiple of 5/10• Identity (0 or 1)	<ul style="list-style-type: none">• Square• Rectangle• Rhombus• Golden	<ul style="list-style-type: none">• Isosceles• Equilateral• Right Triangle

4 5

Survey Instrument

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	Numbers	Parallelograms	Triangles
Mathematical Properties	<ul style="list-style-type: none">• Prime• Perfect square• Power of 2 or 10• Multiple of 5/10• Identity (0 or 1)	<ul style="list-style-type: none">• Square• Rectangle• Rhombus• Golden	<ul style="list-style-type: none">• Isosceles• Equilateral• Right Triangle
Additional Everyday Properties	<ul style="list-style-type: none">• Relative magnitude (small or large)	<ul style="list-style-type: none">• Size (small, large, “skinny”)• Orientation (standard, non-standard, left-leaning)	<ul style="list-style-type: none">• Size (small, large, “skinny”)• Orientation (standard, non-standard)

Analysis

- **Mixed-effects regression models** (Snijders & Bosker, 1999)
 - Participant & Item as random effects
 - Context (mathematical or everyday)
 - Domain (number, triangle, parallelogram)
 - Expertise (middle school student, mathematician)
 - Mathematical and everyday properties
 - e.g., small size, right angle

Results

1) Do middle school students and mathematicians have distinct notions of everyday and mathematical typicality?

- Middle school students tended to rate items that **typical in their everyday life** as also being **mathematically typical**
 - Equilateral triangles are common in their everyday lives
 - Mathematical properties that hold for equilateral triangles are **more likely** to generalize to all other triangles

Student - Triangle

Math (Typical)

1.5

1

0.5

0

-0.5

-1

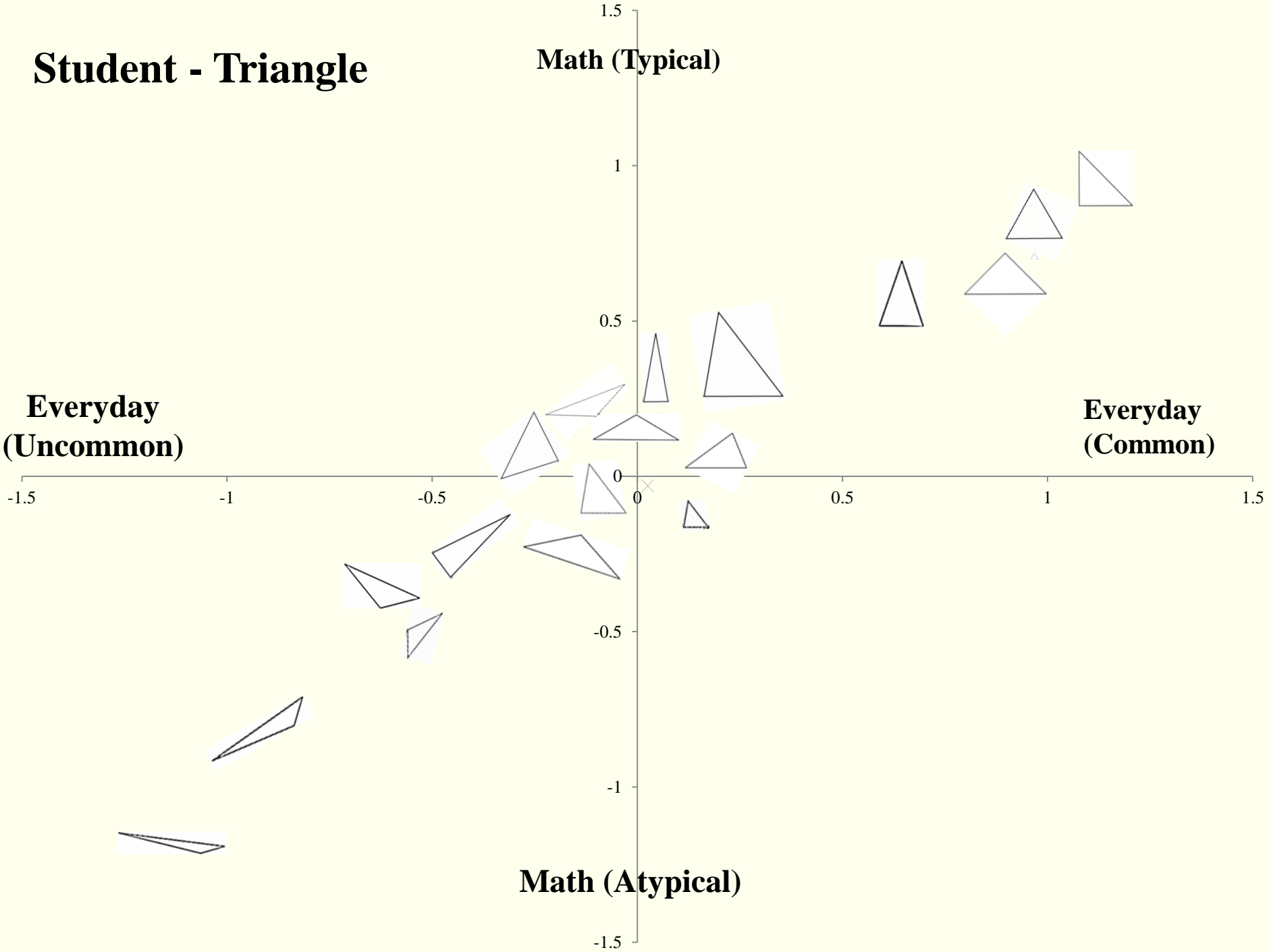
-1.5

Math (Atypical)

Everyday
(Common)

Everyday
(Uncommon)

-1.5 -1 -0.5 0 0.5 1 1.5



Student - Parallelogram

Math (Typical)

1

0.8

0.6

0.4

0.2

0

-0.2

-0.4

-0.6

-0.8

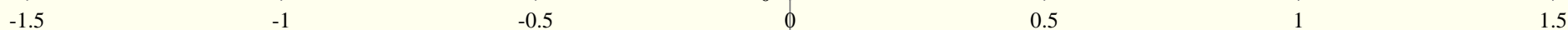
-1

Everyday
(Uncommon)

Everyday
(Common)

-1.5 -1 -0.5 0 0.5 1 1.5

Math (Atypical)



Student - Number

**Math
(Typical)**

25 + 10

0.5

100²

15

1

10000

8

6

18

12

0

**Everyday
(Common)**

**Everyday
(Uncommon)**

-1.2

-0.6

0

0.6

1.2

14

21

900

85

+ 26

84

38

83

- 39

101

102

57

13

Math (Atypical)

-1

Results

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1) Do middle school students and mathematicians have distinct notions of everyday and mathematical typicality?

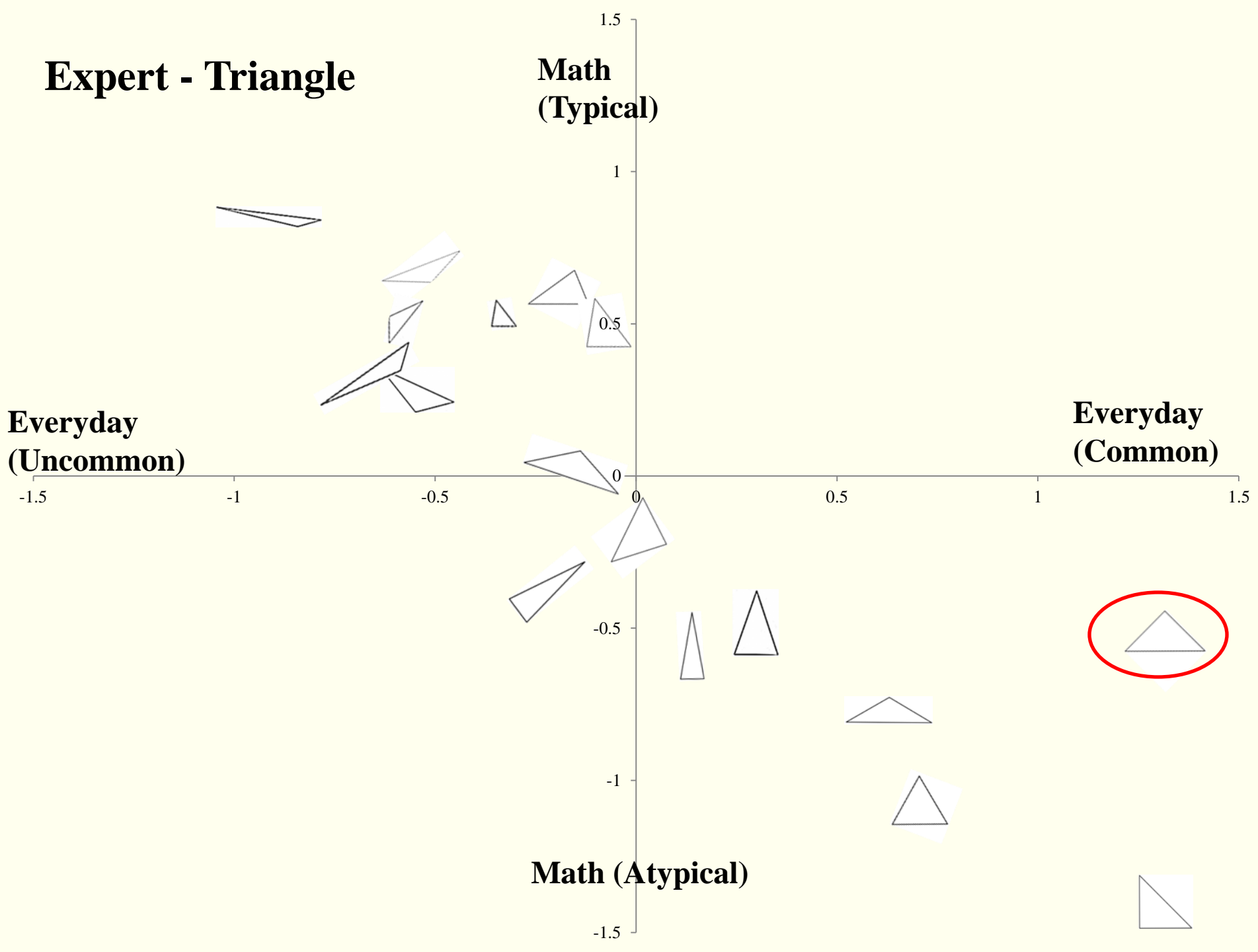
- Mathematicians tended to rate items that were **typical in their everyday life** as also being **mathematically atypical**
 - Equilateral triangles are common in their everyday lives
 - Mathematical properties that hold for equilateral triangles are **less likely** to generalize to all other triangles

Expert - Triangle

**Math
(Typical)**

**Everyday
(Uncommon)**

**Everyday
(Common)**

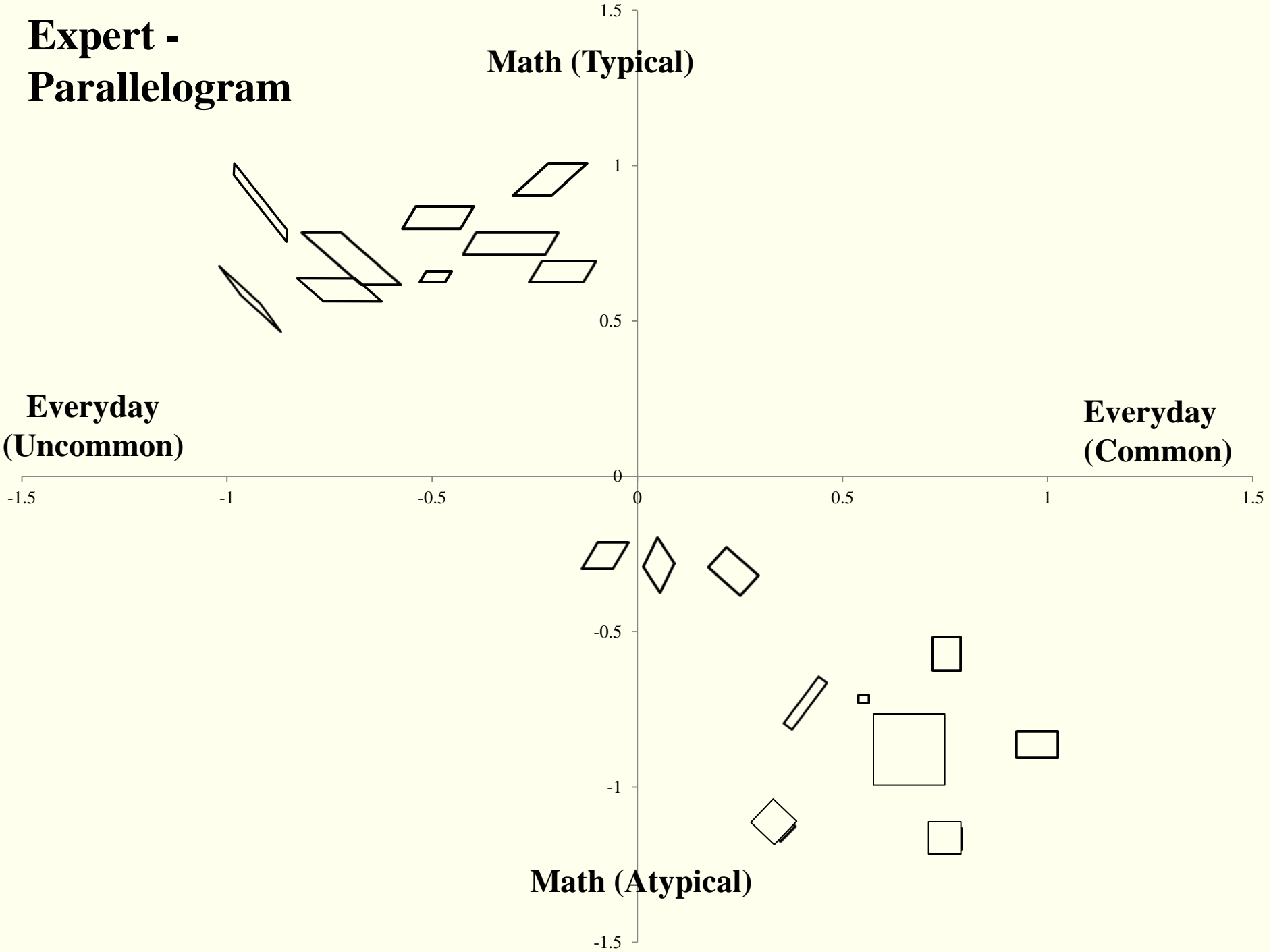


**Expert -
Parallelogram**

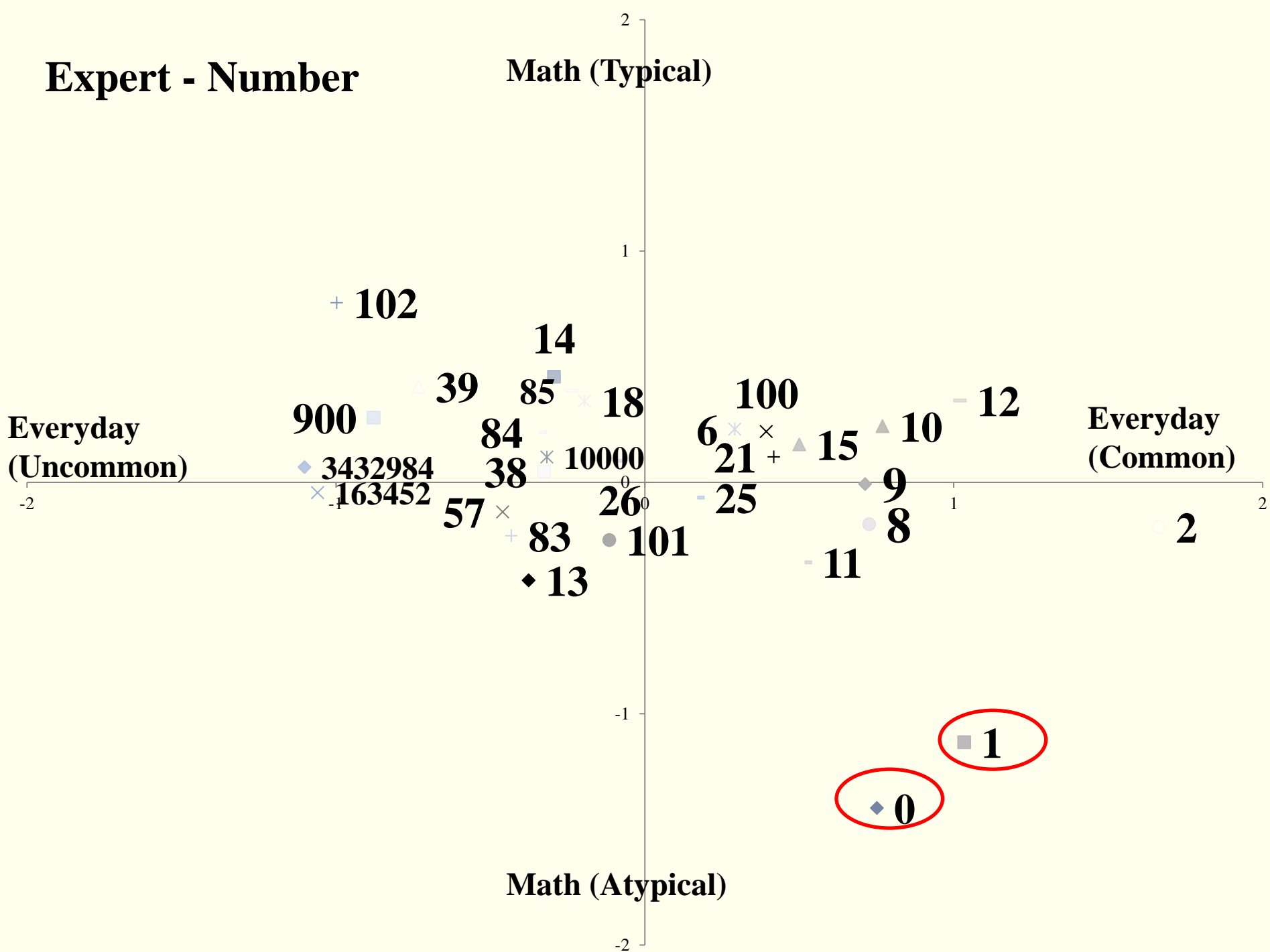
Math (Typical)

**Everyday
(Uncommon)**

**Everyday
(Common)**



Expert - Number



Results

*2) Do middle school students and mathematicians use typicality **strategically** when considering mathematical conjectures?*

- Mathematical properties decrease mathematical typicality*
- Superficial properties not impact mathematical typicality*
- Middle school students showed occasional evidence of strategic thinking
- Mathematicians showed consistent evidence of strategic thinking

Results

Middle School Students:

- Superficial characteristics (size, orientation) impacted *everyday* typicality significantly more than *mathematical* typicality
- Some special mathematical properties, like prime-ness, being an identity, being a square/rectangle/rhombus, being equilateral/right *increased typicality significantly more* in an everyday context than in a mathematical context

Results

Mathematicians:

- Superficial characteristics like size and orientation tended to *only* significantly impact everyday typicality ratings
- Many of the special mathematical properties significantly *decreased* typicality in a mathematical context. Ratings for items with these properties were significantly higher in an everyday context.

Summary

- Middle school students often conflate mathematical and everyday typicality
 - Objects common in everyday life are good candidates for generalization
- Mathematicians separate mathematical and everyday typicality
 - Objects common in their lives often have special mathematical properties that limit generalization

Summary

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- Middle school students have emerging intuitions about mathematical typicality
 - e.g., many recognized 0 and 1 as atypical

“One is sort of unusual I guess. Because it's an origin number of like the units. It's sort of like the center point. So I'd kinda consider that unusual. And because it's used in base ten a lot.”

- Mathematicians have these intuitions consistently, and report using them strategically when exploring conjectures

Implications

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- Students have difficulty reconciling mathematical views of numbers and shapes with everyday experiences
- Expertise characterized by flexible application of formal mathematical knowledge **and** everyday experience, based on the features of problem and context
- Mathematicians are able to move flexibly between mathematical and everyday viewpoints, strategically use examples and typicality

Implications

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- Students should reflect on how mathematical objects are considered differently in the mathematics classroom compared to day-to-day life
- Strategic use of examples and typicality may be important in helping students think more critically about mathematical evidence
- May help students make generalizations about why mathematical conjectures hold, which ultimately supports deductive reasoning and formal proof

The Case of 1729

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- Experts have a dynamic and mathematically-rich ways of seeing and making sense of the world
- Encourage K-12 students to adopt these important perspectives when considering and solving mathematics problems



For More Information...

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- **Saturday from 8:30 – 9:10:** Middle School Students' Example Use in Conjecture Exploration and Justification (Amy Ellis)
- **Saturday from 9:20 – 10:** Stronger Arguments Within Inductive Generalization in Middle School Mathematics (Jennifer Cooper)
- **Saturday from 10:30-11:10:** A Framework for Mathematicians' Example-Related Activity When Exploring and Proving Mathematical Conjectures (Fatih Dogan & Elise Lockwood)

References

- Alcock, L. J. (2004). Uses of example objects in proving. In M. J. Hines & A. B. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 2)*. Bergen, Norway: Bergen University College.
- Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24(4), 359- 387.
- Cooper, J., Walkington, C., Williams, C., Akinsiku, O., Kalish, C., Ellis, A., & Knuth, E. (2011). Adolescent reasoning in mathematics: Exploring middle school students' strategic approaches to empirical-based justifications. In *Proceedings of the 33rd Annual Conference of the Cognitive Science Society*. Boston, MA.
- Feeny, A., & Heit, E. (2007). *Inductive reasoning: Experimental, developmental, and computational approaches*. Cambridge: Cambridge University Press.
- Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *Issues in Mathematics Education*, 7, 234-283.
- Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, 31(4), 396-428.
- Knuth, E., Kalish, C., Ellis, A., Williams, C., & Felton, M. (2011). Adolescent reasoning in mathematical and non-mathematical domains: Exploring the paradox. In V. Reyna, S. Chapman, M. Dougherty, & J. Confrey (Eds.), *The adolescent brain: Learning, reasoning, and decision making*. Washington, DC: APA.
- Osherson, D.N., Smith, E.E., Wilkie, O., Lopez, A., & Shafir, E. (1990). Category-based induction. *Psychological Review*, 97, 185-200.

Survey Instrument

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- 22 parallelograms, 22 triangles, 27 numbers
 - Varied whether items had special mathematical properties (e.g., square, equilateral triangle)
 - Varied common-ness in everyday life
- Many items that are common tend to also have special properties (e.g., 0 and 1)
 - Include uncommon items with special properties, and common items with no special properties
 - Selections based heavily on previous work