Expanding Notions of "Learning Trajectories" in Mathematics Education

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Background

- *Learning trajectories* (LTs) elucidate how students might engage with, reflect on tasks, and develop knowledge through work on those tasks; describe bottom-up natural emergence of students' mathematical knowledge
- *Learning progressions* (LPs) take a top-down approach, focusing on benchmarks for student achievement
- Use of LTs/LPs ranges widely both in breadth and depth: from studies of individual student learning of a single concept, to trajectories covering the recently released Common Core State Standards for Mathematics (2010)

3.NF.1 Understand a fraction a/b as the quantity formed by a parts when a whole is partitioned into b equal parts.
4.NF.4a Understand a fraction a/b as a multiple of 1/b. For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4).
5.NF.5b "Interpret multiplication as scaling (resizing), by explaining why multiplying a number by a fraction greater than 1 results in a product greater than the given number.

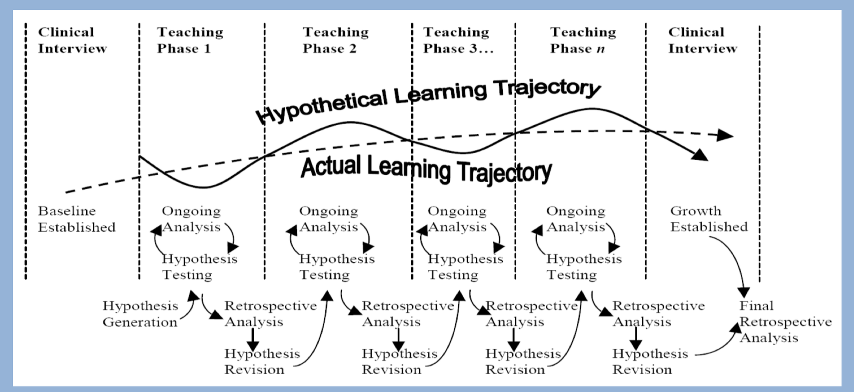
Statement of Problem

- LTs and LPs typically stem from a ***radical constructivist*** perspective on the nature of learning
- What would a LT look like if other theoretical perspectives were taken into account? How can other perspectives inform our understanding of how students gain mathematical knowledge?

The Hypothetical Learning Trajectory

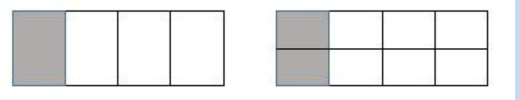
HLTs have 4 characteristics (Simon & Tzur, 2004):

1) Generation of an HLT is based on understanding of the current knowledge of the students involved.
2) An HLT is a vehicle for planning learning of particular mathematical concepts.
3) Mathematical tasks provide tools for promoting learning of particular mathematical concepts and are, therefore, a key part of the instructional process.
4) Because of the hypothetical and inherently uncertain nature of this process, the teacher is regularly involved in modifying every aspect of the HLT.



Vignette

Sarah's fourth grade class is learning about models for equivalent fractions when one denominator is a multiple of the other denominator (CCSM 4.NF.A.1). While the class is discussing the notion of a fraction as a part of a whole, Sarah draws a rectangular diagram to represent 1/4 using 4 equal-sized pieces, one of which is shaded (see image below). The teacher than asks the class to determine how many eighths are in ¼. Sarah divides each of her four subsections in half to produce eight equivalent sections. Paul, having done the same thing as Sarah, raises his hand at tells the teacher that there are two-eighths in one fourth. Sarah agrees because she now has two pieces shaded instead of one piece, and has eight, equal total pieces, which fits with Paul's description of two eighths. Building on this action, the teacher wants to understand if the students see the fractions as equivalent. She asks if one-fourth and two-eighths represent the same amount. Sarah is unsure if this is true since her picture changed from four to eight equal pieces. Other members of the class explain that the same amount is shaded each time, and that adding extra subdivision lines does not change the amount, so they are equivalent. Sarah nods, thinking to herself that the equivalency of fractions is not affected by how many divisions are made to a diagram.



The teacher then asks the class to solve the following problem: 5/9 = ?/90. Her intention is that the students will consider 90 to be too large a number of divisions to make, and will push them to think without a physical diagram. Sarah overhears her friend Joe say the answer is 50 because the 5 shaded pieces will be divided into 10 parts each. Sarah looks confused, but tells Joe that each of the 9 original pieces will need to be divided into 10 parts to get 90, so each of the 5 shaded pieces will be divided into 10 parts to get 50. Joe raises his hand and announces to the class what they've discovered. Sarah tells the class that they are the same thing because the same amount of the rectangle would be shaded each time but the name of the fraction differs, helping her confirm her original idea that the number of divisions does not affect equivalency.

Next, the teacher gives the class the following prompt: "Now solve 5/9 = ?/18 and 5/9 = ?/54 without using diagrams. Describe a general procedure you can use to find equivalent fractions like this." Sarah thinks each piece will be cut into two pieces, so the numerator would be 10. In the second case, each piece would be cut into 6 pieces, so the numerator would be 30. After looking at her answers to these two problems and to the prior task, she writes that: "You can generate equivalent fractions by figuring out how many pieces each piece has to be cut into." The teacher then asks how one would know how many pieces it has to be cut into. Joe raises his hand and notes the necessity of comparing the two denominators to determine the correct cut. He says "if it's 9 and you need 18, you have to cut in half". The other students, including Sarah, express their agreement with Joe. Sarah then replaces 9 and 18 with x and y. She notes that if y is 3 times as large as x, then the numerator above y will need to be three times as large as the numerator above x. In her head, she compares this to having a diagram with x equal divisions, and dividing each of those x divisions in into three equal divisions.

Vignette from Multiple Perspectives

What is Sarah really "learning"?

- **Cognitive Science:** Learning as the acquisition and modification of *knowledge components* (KCs), acquired units of cognitive function; "gold standard" for learning is acquisition of abstractions and strategies that are highly general, portable, and easily retrieved
- **Radical Constructivism**: Learning as the development of viable interpretations of experience based on the reflexive processes of assimilation and accommodation; subtle shifts in understanding that modify internal schemes of concepts
- **Situated Learning**: Learning as trajectory of participation practices that are distributed across person, time, place, and activity; examine the development of learner identities and community norms

Principle	Cognitive Science (CS)	Radical Constructivism (RC)	Situated Learning (SL)
1	Generation of the HLT takes into account both current knowledge states of students (i.e., KCs acquired), as well as theoretical and empirical models of expert and novice performance (from cognitive task analyses). This trajectory is then used as a guidebook on how to move learners from novice to expert performance. Also considered is the complexity and interactivity of the knowledge components in question.	HLTs focus on creating a model of what a student knows without considering that model as "correct" because that determination is not within the bounds of radical constructivism. These models typically focus on an "epistemic" student, one to whom certain ways of thinking are assumed. The researcher uses existing literature and/or works with a sample of students to identify schemes that appear common. This model helps predict assimilations and accommodations that may occur during the learning process.	When developing a HLT conjectures about the collective development of the mathematical community are taken into account by focusing on the practices that are currently taken-as-shared in the classroom. These practices are a starting point for new norms to develop. These new norms may be those that are shared by the larger community of mathematics practitioners. However, "school mathematics" represents its own system whose norms, standards, and practices are different from problem solving in other contexts.
2	HLTs focus on planning out sequences that will allow learners to grasp flexible, portable, mathematical abstractions that transfer, are retained over time, and prepare for future learning (i.e., they focus on robust learning outcomes). Here the learning of the particular concepts is not everything – the planned HLT should allow for transfer and seek to prepare students for future learning by promoting productive metacognitive behaviors.	HLTs build from students' initial schemes to propose and evaluate how those schemes develop to various levels of sophistication. The development takes into account assimilations and accommodations to explain the fine-grained changes in students' knowledge as well as more global changes in their understanding.	HLTs build from initial group practices to take into account that practices will evolve through participation in activity. HLTs involve a trajectory of participation practices as a member of a social system, or the ways in which participation in activity contributes to growth as a learner and future participation in other activity systems of value to the learner.
3	Mathematical tasks should be designed to promote and assess different types of learning: memory/fluency, induction/refinement, and understanding/sense-making. This perspective also accentuates the time cost of different instructional methods, noting important tradeoffs between amount of learning and instructional time spent.	Mathematical tasks are intended to foster the development of existing schemes, as they develop to form an HLT. In some cases, students' schemes will be sophisticated enough that accommodation need not take place. In others, the task creates a need for a change in existing schemes.	Mathematics tasks act as tools that support the emergence of participation practices that would be based on increasingly sophisticated ways of acting on mathematical ideas. Communities have the power to shape what counts as mathematics, including the meaning of terminology, concepts, and principles, and how they can be applied.
4	The teacher's role is not described in this framework, as it is intended to be applicable to artificial intelligence systems. However, generally, the role of the teacher is to facilitate student's acquisition of the KCs that represent the domain of interest. The teacher facilitates the given trajectory of KC development, while also using common sense and professional experience to adapt to any specific issues that arise that the trajectory cannot address.	The teachers' role is to act as a facilitator that supports the use and discussion of tasks as students' schemes for a mathematical idea develop. The teacher is sensitive to students' in the moment sense making, and makes decisions on instruction based on that emergent sense-making.	The teachers' role is to act as a facilitator and knowledgeable community member that supports the use and discussion of tasks as community practices for a mathematical idea develop. Learners and teachers co-construct classroom social practices through interaction. A "community of validators" critically examines claims through argumentation rather than through appealing to authorities like the teacher or text.

Discussion and Implications

- Validation-oriented (CS) versus evolutionary (RC, SL) LTs involve different theoretical underpinnings
- Knowledge as individual (CS, RC) vs. distributed (SL) has important implications for LTs that can be built
- Range of educational outcomes – including practices, identities, metacognitive strategies (SL, CS) - should be considered in LTs
- Teacher's role ranges from driver of LT (RC), to passive implementer of pre-determined LT (CS), to facilitator of emergent LT (SC)
- Important to incorporate the varying learning perspectives into the discussion of LTs, and to explicitly acknowledge how the theoretical perspective being taken influences LTs

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