Grounding Mathematical Justifications in Concrete Embodied Experience: The Link between Action and Cognition

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Purpose

A central issue in mathematics education relates to *grounding*, or the question of how abstract, mathematical ideas can be connected to students' prior knowledge and experience, such that their meaning becomes connected with concrete, perceptual referents that can be readily understood (Goldstone & Son, 2005; Koedinger, Alibali, & Nathan, 2008). Recent research on grounding interventions in mathematics classrooms has shown that tying mathematical formalisms to students' concrete, everyday experiences can support their long-term learning of mathematical ideas (Walkington & Sherman, 2011; Walkington, under review), suggesting great promise for these approaches. One method of grounding that has recently received attention in mathematics education is participation in body-based experiences where students engage in physical interaction with the world, and mathematize their experiences (e.g., Abrahamson & Howsin, 2010; Bolger et al., 2010; Moses & Cobb, 2001; Nathan, 2008).

This research draws on embodied views on the nature of cognition, which describe cognitive processes as intimately tied to perceptual and motor systems in the body (Wilson, 2002). In the area of mathematics particularly, embodied theories stand in stark contrast to the "romantic view" of mathematics as an amodal, transcendental, objective feature of the universe. Instead, embodied theories give rise to a vision of mathematics as a domain that is constructed of body-based experiences of human beings with the world (Lakoff & Nunez, 2007). This theoretical perspective suggests that embodied experience should help students to understand and construct mathematical meaning, and that such experience could in fact be critical to the learning of mathematics. However, until recently, research in mathematics education had not begun to deeply explore the ways in which students can learn with their bodies.

Recent research in mathematics education and learning science has examined how students understand mathematical ideas through perception and action, and has demonstrated that students can learn foundational mathematical ideas through embodied activities. For example, understanding of number is spatial and tied to bodily orientations (Dehaene, Bossini, & Giraux, 1993), and children approach arithmetic problems using modeling approaches where they manipulate objects or count with their fingers before moving to mental number facts (Carpenter & Moser, 1984). When working with arithmetic and algebraic equations, students perceive symbols and equations as having concrete, spatial and perceptual qualities (Landy, Brooks, & Smout, 2012). When learning fractions, actions coupled with interpretations serve as developmental precursors to general mathematical procedures, which can later be enacted mentally (Martin, 2009; Martin & Schwartz, 2005). Providing students with physically-grounded perceptual experiences can help them understand core ideas of proportionality (Abrahamson & Howsin, 2010), and physical and simulated action can support arithmetic and algebra story problem-solving (Glenberg et al., 2007; Nathan, Kintsch, & Young, 1992). Thus embodied activities seem to have great potential in mathematics education to draw upon students' intuitive ways of understanding phenomena, and provide a powerful body-based system for reasoning about novel ideas.

Previous work in mathematics education that leverages embodied theories has focused heavily on students' understanding and learning of topics like number and algebra. Here, we focus on how embodied activities may support students in understanding mathematical conjectures, and engaging in practices of mathematical justification. Justification and proof are challenging practices for students to master as they reach secondary mathematics classes that more heavily emphasize this type of mathematical thinking, especially high school geometry (Healy & Hoyles, 2000; Hoyles & Healy, 1999, 2007). Research has shown that students may test examples rather than engaging in general justification (Knuth et al., 2002), and rely on description and perception rather than formal mathematical reasoning (Jones, 2000). Students can find deductive proofs unconvincing (Chazan, 1993), with the ubiquitous two-column proof in geometry sometimes taught in such a way that students fail to appreciate its utility (Harel & Sowder, 1998).

One important method to support conceptual understanding of mathematical conjectures using action is through Dynamic Geometry Systems (DGS). Systems like Geometer's Sketchpad (GSP) allow students to engage in action-based manipulations of mathematical objects on a screen in front of them, in order to support students in understanding and exploring mathematical conjectures (Christou et al., 2004; Marrades & Gutierrez, 2000). The dynamic, action-based understanding that is supported with such systems may provide a critical resource for mathematical reasoning. However, these systems can be difficult to implement in classrooms as a tool to support high-level mathematical tasks, as the mechanics and affordances of the systems can be challenging for students and teachers to adopt (Sherman, 2012). Here, we explore the implications of having students perform action-based manipulations with their own bodies, rather than enacting these actions on objects or representations in a dynamic environment. We will examine how action and cognition are intertwined as students generate mathematical justifications for conjectures in geometry and number theory, after performing actions that are relevant or irrelevant to a key insight behind the conjecture's justification.

Theoretical Framework

Action and Cognition

Embodied views on the nature of cognition posit that all mental processes are rooted in perceptual and motor systems (Wilson, 2002). This implies that mental representations of objects, including mathematical objects, are experiential, perception-based, and multimodal in nature (Barsalou, 1999; Lakoff & Nunez, 2000; Landy et al., 2012), and that there may be an important interplay between action and cognition during mathematical problem solving. For example, Glenburg et al. (2007) propose that language is interpreted by making connections between words and phrases and perceptual representations of objects, and then simulating actions on those objects. Accordingly, in a study of children solving mathematics story problems, they found that when students engaged action or simulated action with story-relevant objects, their problem-solving performance was improved. Similarly, Nathan et al. (1992)

found that when students are given support in simulating the action of algebra story problems through computer-based animations, they make stronger connections between symbolic representations and story scenarios. Thus action and simulated action are intimately tied to problem-solving, with mathematical ideas grounded in multimodal representations.

Recently, researchers have begun to test the hypothesis that action has a direct impact on cognition – i.e., that action influences thought. One idea is that body-based actions may affect reasoning about objects by integrating perceptual and motor experience into mental representations of those objects (Goldin-Meadow & Beilock, 2010). Action may also influence thought by facilitating related or residual activations in working memory (Thomas & Lleras, 2009). Research on directed movement using classic "insight" problems[1] has supported the idea that action has a direct and implicit effect on cognition. For example, Thomas and Lleras (2007) showed that directing learners to move their eyes in accordance with the solution to a well-known insight problem (Duncker's radiation problem) resulted in more correct solutions. Similarly, Thomas and Lleras (2009) showed that directing learners to move their arms in accordance with the solution of another insight problem (Maier's two-string problem) increased problem-solving success. What was significant about both of these studies is that most participants were not consciously aware of the relationship between the actions they were taking with their bodies and the problems they were solving. It was hypothesized that action had an implicit effect on cognition, that body movement can guide higher-level cognitive processes even if the learner is not aware of a connection. However, an important unanswered question is how action can direct *mathematical* thinking in *educationally productive* ways. To explore this idea, we look to research on gesture in instructional settings.

Gesture as a Scaffold in Instructional Settings

Gesture can be considered as an important type of action (Goldin-Meadow & Beilock, 2010) which has been theorized to emerge from embodied perceptual and motor simulations that underlie

[1] i.e., Problems that require solvers to overcome an impasse in their reasoning, and move beyond a naïve or immediately obvious approach. See Thomas & Lleras (2007; 2009)

mental imagery and language processing (Hotstetter & Alibali, 2008). Research in instructional contexts suggests that gestures play an important role in student learning. Teacher gestures can scaffold student comprehension, with representational gestures used to ground abstract mathematical ideas in body-based form (Alibali & Nathan, 2007; Nathan, 2008). Gestures can also guide student attention and communicate spatial, relational, and embodied concepts (Alibali, Nathan, & Fujimori, 2011). Finally, gestures can serve to link ideas and representations, with gestural catchments (i.e., repeated representational gestures, see McNeill & Duncan, 2000) creating structural mappings between different entities to show relatedness (Alibali et al., 2011; Nathan, 2008).

Recent research has also begun to explore how performing gestures can influence the gesturer's thought processes (Goldin-Meadow & Beilock, 2010; Goldin-Meadow, Cook, & Mitchell, 2009). For instance, requiring students to use their body to represent ideas through gesture supports long-term retention of mathematical concepts (Cook, Mitchell, & Goldin-Meadow, 2008; Gerofsky, 2010), and teaching students to gesture can also instigate the creation of novel mathematical ideas (Goldin-Meadow et al., 2009). When children were taught gestures relating to a new approach for balancing equations without a verbal explanation of the strategy, many were able to subsequently add this strategy to their repertoire. These children also learned more than children asked to produce partially correct gestures relating to the strategy, or children asked to produce no gestures (Goldin-Meadow et al., 2009). Thus gestures are a form of body-based action that may be a particularly important scaffold for learning in the mathematics classroom.

Gesture and Action as Cohesion-Producing Mechanisms

We conducted video observations of a variety of project-based high school engineering ($N = 25$) and geometry ($N = 17$) classes, and uncovered several ways in which teachers and students use their bodies, particularly gesture, to build and promote mathematical understanding. We found that representational gestures are often used to re-invoke absent objects or representations from previous or future stages of project work, while pointing gestures are used to create explicit mappings or links between co-present modalities or representations. Beat gestures can be used in conjunction with speech to emphasize important mathematical concepts as they arise across different contexts (see McNeill, 1992, for gesture types). These types of gestures often co-occur with an instructional move we call *projection* (Nathan et al, under review), where a teacher or student explicitly references a past or future activity the class will be or has been engaged in, in order to connect mathematical ideas as they develop and transform over time and space. For example, a teacher might explicitly connect a deductive proof the class is developing on the board to a prior exploration of the conjecture in a DGS. In these ways, teachers and students use language and gesture to build *cohesion* of mathematical ideas across the complex and shifting modalities and temporalities that arise in project-based settings.

One particularly interesting series of interactions from our observations in geometry involved a set of lessons about angles and shapes inscribed in circles. The classroom context alternated between the computer lab, where the students inductively explored geometric theorems by constructing objects in GSP, and the classroom, where the teacher supported students in developing deductive proofs through discussion. In both contexts, the teacher made use of representational gestures to ground abstract geometric ideas in body-based form, and deictic gestures to focus student attention on salient mathematical features of different representations. For example, in the laboratory when students were using GSP to prove that opposite angles in an inscribed quadrilateral are supplementary, the teacher first identified the relevant angles and arcs on students' GSP computer screen using gesture (Figure 1, left), and then later traced two arcs of an imaginary circle in the air, and then traced all the way around the circle to show that these two arcs together form the entire circle (Figure 1, right). This gesture was repeated again later as a gestural catchment (McNeil, 1992) during a whole-class discussion of the deductive proof of the conjecture. Thus the teacher used the instructional move of projection in conjunction with gesture to create an action-based mapping between the different contexts in which learning was taking place (Nathan et al., under review).

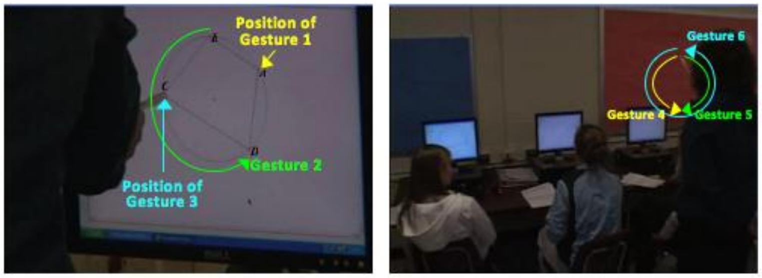


Figure 1. High school geometry teacher uses representational and deictic gestures to support students' understanding of inscribed quadrilateral conjecture

Movements like those the teacher is making in Figure 1 may be important resources when considering how to provide students embodied experiences with mathematics. These gestures were enacted by the teacher when it was clear that students were struggling with the central concepts in the lesson. Indeed, throughout these lessons and others, we observed many instances of teachers (and sometimes students) using their whole bodies to build cohesion of mathematical and scientific ideas across disparate contexts, using action, manipulation and virtual manipulation of objects, and representational, deictic, and beat gestures (Nathan et al., under review; Walkington et al., in press).

To frame the study presented here, we present new analyses from our classroom research that informed our study design. Figure 2 shows an excerpt from the same series of geometry lessons where the teacher and a student use iconic gesture to come to a mutual, embodied understanding of the meaning of the term "intercepted arc." The teacher makes her hands into a "V" (lines 3, 5, and 9) in order to correct a student's confusion over the terms "intercept" and "intersect." The student seems to understand the difference between the terms as he differentiates them with two distinct hand gestures (line 6).

Figure 2 illustrates how grounding through gesture helps to resolve a language specific confusion between two similarly sounding math terms, and thereby helps a student make sense of the phenomenon at hand. The body movements enacted by the teacher and the student are key to their mathematical communication, and to the construction of mutual understanding of the geometric idea. Such instances make it clear that gesture and action are critical modalities when considering how to support students' mathematical reasoning and communication, as they spontaneously arise during key instances where conceptual understanding must be built. Classroom episodes like the two discussed here led us to believe that we could design action-based interventions where students could move their bodies to support their understanding of mathematical ideas, and thus more successfully engage in practices of mathematical justification.

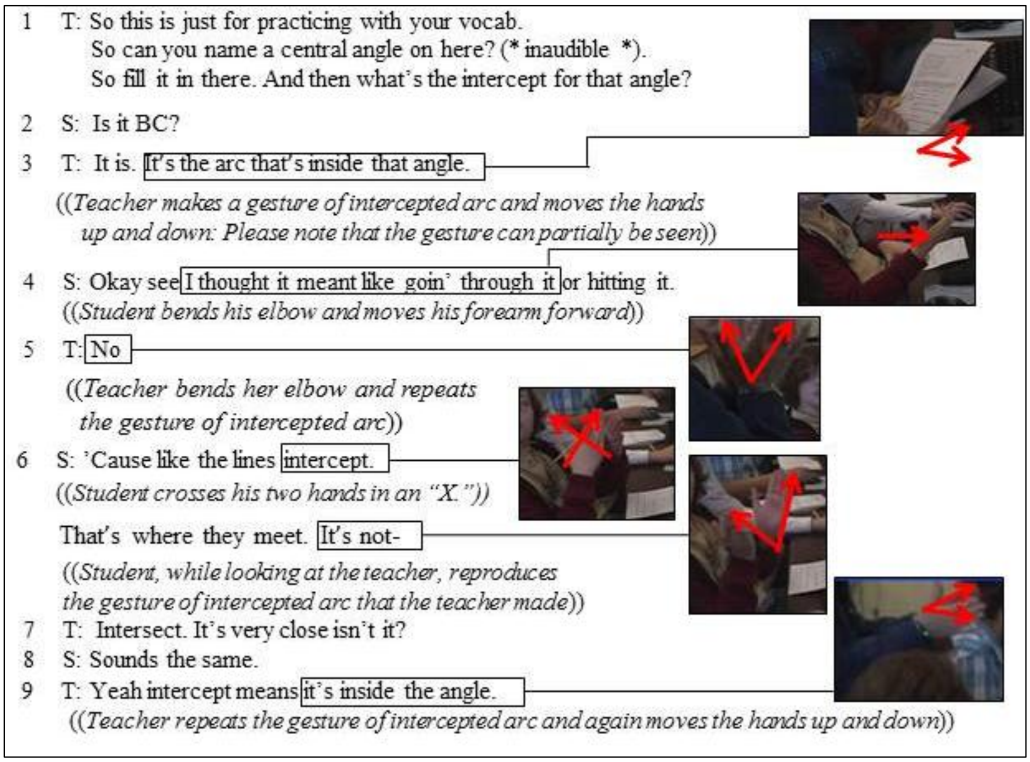


Figure 2. Teacher and student come to mutual understanding of inscribed angles and intercepted arcs using representational gestures

Character and Observer Viewpoint Gestures

One important distinction when considering the impact of gesture and action on cognition and learning is character-viewpoint versus observer-viewpoint gestures (McNeill, 1992). In character-viewpoint gestures, the gesturer is directly taking the role of an actor or agent inside the story; for instance, the gesturer might show the movement of a described character's hands by moving their own hands. In observer-viewpoint gesture, the gesturer remains at a distance from the described action in the story, acting as an outside spectator; for instance, the gesturer might show a character's path by representing the entire character with their moving hand.

In our classroom research, we also saw powerful examples of teachers initiating change in viewpoint to support students' mathematical understanding. For example, we observed one teacher who was describing the idea of a "plane" to his geometry class by showing them a diagram of a parallelogram on his whiteboard. A student expressed his confusion over the term, and the teach enacted a change of viewpoint and began discussing how the walls and floors of the classroom around them were planes, using his body to enact sweeping representational and deictic gestures (Srisurichan et al., under review). However, such changes in viewpoint are far from common; in classrooms, mathematics is often experienced at an arm's length, as an outside observer looking onto a computer screen or a textbook, rather than through close-up first-person experience.

Gerofsky (2010) argues that character viewpoint gestures may be especially important to the learning of mathematics, as they represent close-up, embodied experience with the world. A study of high school students showed that students identified by their teachers as mathematically-able were more likely to use character-viewpoint, rather than observer-viewpoint, gestures when discussing functional graphs – i.e., they gestured the shapes of graphs with their entire bodies. Using this notion that mathematically competent students are those whose gestures involved "being the graph" (p. 331), Gerofsky designed a classroom intervention where students were given multi-modal, embodied experiences with graphs that encouraged this first-person perspective and a character viewpoint mode of gesturing. Gerofsky found that a year later, students had excellent retention of the properties of graphs they learned in this manner, with many referring to the movements and gestures they had performed. However, it is not clear if character-viewpoint gestures are always the best method of supporting students in learning mathematical ideas. Goldin-Meadow and Beilock (2010) suggest that observer viewpoint gestures that strip away some of the perceptual details of the action may better promote generalization and transfer. In the study to be presented here, viewpoint will be an important factor when considering how to design embodied activities to support student learning.

Research Questions

Here we seek to explore the idea that action can guide and direct mathematical thought in ways that are productive to the development of deep, conceptual, embodied understanding. Our overarching research question for this program is: How does body-based action and gesture relating to mathematical conjectures influence reasoning about and conceptual understanding of mathematical justifications? More specifically, we will use the data collected from our studies thus far to address the following research questions:

1) **How are action and gesture used *spontaneously* to support mathematical justification?**

Previous research has examined how gesture is used by teachers as a scaffold in instructional settings, but has not deeply explored the role of gesture in reasoning about and communicating mathematical justifications. Based on our classroom research, learners use gestures and their bodies to support their reasoning when considering novel or challenging mathematical ideas. Learners also use gesture and action as a critical modality to communicate convincing, understandable, and generalizable arguments to an external audience. Here, we will analyze the ways in which participants use their bodies spontaneously when asked to engage in mathematical justification.

2) **Is there an *implicit* link between action and cognition that can support mathematical justification?**

Prior work has suggested that action can have an implicit effect on cognition – that actions can produce thoughts or novel insights, even if the learner is not consciously aware of the connection. However, little research has examined whether action can be used to produce important *mathematical* insights, especially in the area of justifying conjectures and explaining how mathematical phenomena work. Here, we will examine how directing participants to perform relevant gestures impacts their construction and communication of mathematical justifications, when they are not aware of a connection between the relevant gesture task and the mathematical task.

3) **How can *explicitly* linking action-based interventions to mathematical ideas support mathematical justification?**

Our previous work in geometry and pre-college engineering classrooms has suggested that *projection* – a move where a teacher or student uses language and gesture to link instantiations of a mathematical concept that arise over different classroom encounters – is a powerful tool to produce and maintain the cohesion of ideas over time. This is because students may otherwise struggle see to how different representations and activities are related, if these critical links are left implicit by the teacher. Although we believe projection is an important scaffold for learning, little research has experimentally examined its impact. Here, we will examine whether explicitly linking relevant performed gestures to mathematical justifications using projection can support participants in engaging in successful practices of mathematical justification.

4) **What is the role of *viewpoint* when using gesture to support practices of mathematical justification?**

When considering the role of gesture in mathematics learning, recent research has revealed that the scale or viewpoint on which the gesture is taking place may be critical to supporting students' understanding. Observer viewpoints involving students perceiving and interacting with mathematical objects in front of them are most often used in the classroom; however, character viewpoints where the student has the opportunity to *be* the object or act *inside* the problem may be especially effective for supporting learning. Here, we examine whether character or observer viewpoint gestures are more effective in supporting participants' construction and communication of justifications.

Methods

We will report results from two phases of data collection. We report qualitative results from our pilot study, where we engaged 47 undergraduate students in interviews where they were asked to provide justifications for 4 of 8 different mathematical conjectures or tasks. We interweave these qualitative results with quantitative results from our current experimental study, where we have engaged 57 undergraduate students in think-aloud tasks (Ericsson & Simon, 1993) where they were asked to provide justifications for 4 mathematical conjectures or tasks.

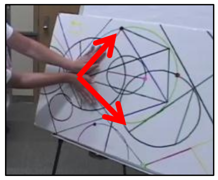
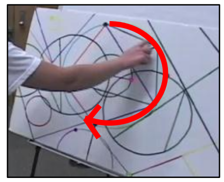
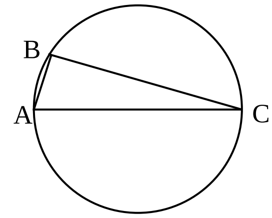
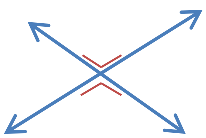
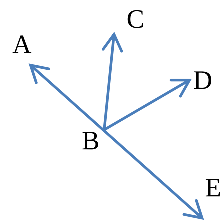
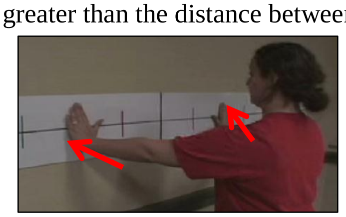
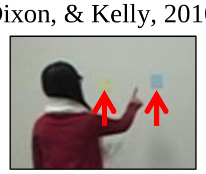
Pilot Study Design

During the pilot study, we interviewed 47 undergraduate students enrolled in a psychology course (students received extra credit for participating in the study), and asked each to provide justifications for two of four different initial tasks. Before being presented with each initial task, the students were asked to perform actions that were either related (relevant) or unrelated (irrelevant) to a mathematical justification for the task. After justifying each initial task, participants were asked to provide a justification for a second related task that was considered to be a "transfer task." Table 1 lists the 4 initial tasks and 4 transfer tasks used in the study, and also gives examples of the relevant and irrelevant actions that participants were instructed to perform before being given each initial task. At the end of each session, participants were asked whether they saw a connection between the actions they performed and the tasks they were asked to justify (it was not suggested to them previously that there was a connection). If they had not, the interviewer made a backwards projection, telling the participant which actions they performed were related to which initial task. The participant was then asked to try to figure out exactly what the connection was. Most participants did not report noticing the connection until the interviewer made the backwards projection.

There were 4 different interviewers, and interviewers were allowed to engage in ad-hoc questioning to explore each participant's mathematical thinking. This type of questioning was not permitted in the experimental study that followed, but was included in the pilot study because it better allows for deep qualitative analyses of participant thinking. Thus our qualitative data are drawn from our pilot study, while performance/accuracy trends are taken from the experimental study. Further, in the pilot study, the conditions and the tasks themselves were fluctuating as a protocol was being developed – for instance some participants were permitted to use pencil and paper while others were not; some participants performed their actions using poster board while others used an interactive whiteboard; etc.

Table 1. Tasks given during pilot interviews

Initial Tasks	**Actions Performed**	**Transfer Tasks**
Inscribed quadrilateral: Mark came up with the	**Relevant Action**: Participants performing relevant actions were asked to move their hands in "V" motions	John came up with the following conjecture: In

following conjecture: Opposite angles in an inscribed quadrilateral are always supplementary (i.e., add up to 180°).	(similar to Figure 2) to represent inscribed angles, and also used their index fingers to do arc tracing motions (similar to Figure 1) on a board with various lines and arcs. This movement was intended to direct them to the insight that the two inscribed angles intercept the entire circle.   **Irrelevant Action**: Other participants traced a random set of lines on the board.	the circle below, if AC is the diameter of the circle, then ABC must be a right triangle. 
Vertical angle: John came up with the following conjecture: Vertical angles are always congruent (i.e., they have the same measurement). 	**Relevant Action**: Participants used fingers or arms to sweep out two sets of supplementary angles. Each set of supplementary angle contained one angle that was shared in both sets, to support the insight that if x+y = 180, and y+z = 180, then x+z = 180.  **Irrelevant action**: Participants used fingers or arms to trace or sweep out angles that did not embody a particular solution.	In the diagram below, if the measure of angle ABD is x°, and angle CBE is y°, how could you figure out the measure of angle CBD? 
Triangle: Mary came up with the following conjecture: For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side.	**Relevant Action**: Participants performing relevant actions were asked to stretch their two arms to try to reach different marks on a horizontal line, some of which were out of reach. This movement was intended to support the insight that they could only reach the two marks if the sum of the lengths of their arms was greater than the distance between the marks.  **Irrelevant Action**: Other participants touched each mark on the horizontal line in sequence.	Jenny came up with the following conjecture: The sum of the lengths of any three sides of a quadrilateral must be greater than the length of the remaining side.
Gear: An unknown number of gears are connected in a chain. You know what direction the first gear turns, how could you figure out what direction the last gear turns? Provide a justification as to why your answer is true.	**Relevant Action**: Participants performing relevant action were asked to alternate tapping with their index finger back and forth on two post-it notes on the wall. This is a well-known gesture made by people who discover the solution to alternating gear problems (Boncoddo, Dixon, & Kelly, 2010).  **Irrelevant Action**: Other participants were asked to repeatedly tap on a single post-it note.	Eleven gears are connected in a circle. Can the gears turn? Provide a justification as to why your answer is true.

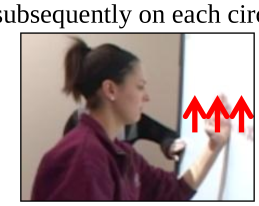
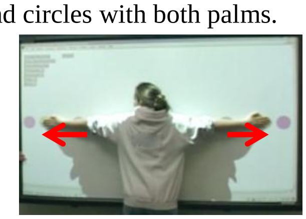
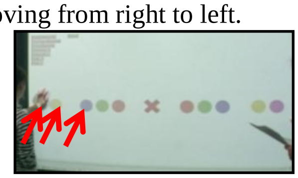
Experimental Study Design

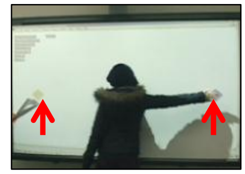
We engaged an additional 57 undergraduate students enrolled in a psychology course in think-aloud tasks where they were asked to provide justifications for the two final tasks in Table 1 – the triangle conjecture and the gear task. These two tasks were chosen because the relevant actions seemed most effective based on the pilot data. Participants again received the associated transfer tasks after each initial task. Before being given each initial task, participants were asked to perform actions that were either relevant or irrelevant to a justification associated with the task. However, this time, the actions were always performed on a large interactive whiteboard that scaled the actions to the participant's body size – this was accomplished by measuring each participant's height, arm span, and hand length using the board at the beginning of the session. There were also two different sets of relevant and irrelevant actions participants could receive for each initial conjecture – actions performed at an observer viewpoint, and actions performed at a character viewpoint. The different action conditions for the experimental study are summarized in Table 2. The visual fields on the whiteboard were always identical for the relevant action and corresponding irrelevant action conditions, and were identical but sized differently for the character versus observer viewpoint conditions. The participants would always perform each sequence of actions a total of 4 times before being presented with the corresponding initial task.

Thirty-seven of the 57 participants received irrelevant actions preceding one initial task, and relevant actions preceding the other initial task. Whether they performed relevant or irrelevant actions first was counter-balanced, as was whether they received the triangle or gear task first. Each participant would be directed to perform character viewpoint gestures for one conjecture, and observer viewpoint gestures for the other conjecture. Whether they performed character or observer viewpoint gestures first was also counter-balanced.

Participants were asked to provide a justification for each task without being told there was any connection between their gestures and the problem. At the end of the interview, participants were asked if they had noticed any connection between the problems and the actions they performed. All participants in the experimental study reported not being aware of any connection, and were then told that there was a connection between the relevant actions they performed and the corresponding initial task. They were asked to try to figure out what the connection was, and were given a second opportunity to provide a new justification for the initial task and transfer task. This second chance was included to explore the role of backwards projection in supporting justification practices, by explicitly connecting an action-based experience to a mathematical task post-hoc. The tasks had been revised based on the pilot study to make the connection as subtle and difficult to explicitly notice as possible, since we wanted to explore whether there was an implicit link between action and cognition.

Table 2. Tasks and action conditions for experimental study

Initial Task	**Observer Viewpoint Actions**	**Character Viewpoint Actions**
Triangle: Mary came up with the following conjecture: For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side. Provide a justification as to why Mary's conjecture is true or false.	**Relevant**: Participant places the side of the heels of both their palms on pairs of circles (same colors) and touches the tips of the middle fingers together. Circles on the screen were scaled to the participant's hand length such that s(he) could not touch fingertips together on the far-end circles. This particular action was chosen because many participants who successfully justified the conjecture in the pilot study spontaneously used these gestures.  **Irrelevant**: Participant places side of heel of left palm subsequently on each circle, then places side of heel of right palm subsequently on each circle. 	**Relevant**: Participant places both palms on pairs of circles (same colors) while keeping the elbows locked and the arms straight. Circles on the screen were scaled to the participant's arm span such that (s)he could not reach the far-end circles with both palms.  **Irrelevant**: Participant taps each circle with their left palm moving from left to right, and then taps each circle with their right palm, moving from right to left. 
Gear: An unknown number of gears are connected in a chain. You know what direction the first gear turns, how could you figure out what direction the	**Relevant**: A yellow and a blue diamond are a hands-length apart on the screen. Participant alternates between tapping blue and yellow diamond using index finger. This is a well-known gesture made by people who discover the solution to alternating gear problems (Boncoddo, Dixon, & Kelly, 2010).	**Relevant**: A yellow and a blue diamond are an arm span apart on the screen. Participant alternates between tapping blue and yellow diamond using palm.

last gear turns? Provide a justification as to why your answer is true.	 **Irrelevant**: Participant repeatedly taps only the blue diamond with their index finger.	 **Irrelevant**: Participant repeatedly taps only the blue diamond with their palm.

The remaining 20 of the 57 participants performed relevant actions for both tasks they received. These participants were told prior to attempting their initial task that there was a connection between the actions they had just performed on the whiteboard, and the justification they were going to be asked to provide. This condition was included to explore the effect of forwards projection, or explicitly connecting action-based experience to a mathematical task before the task is presented. Participants in this condition also received backwards projection at the end of the interview, and were given the second chance to justify both the initial and transfer tasks (the transfer tasks can be found in Table 1).

Five interviewers conducted these think-alouds. All speech by the interviewer was scripted, and many of the instructions were delivered via voice audio files that the interviewer would simply play. Interviewers were not permitted to ask any ad-hoc questions – they had standardized prompts reminding the participant to think aloud and to read the question out loud. They also had standardized prompts reminding the participant that the interviewer could not answer any questions or indicate when the participant should move forwards. Participants constructed and communicated all justifications using verbal language and gesture – no paper was provided. Participants were instructed to press a button on the screen when they felt their justification was complete, and were subsequently prompted to repeat their justification one more time. Each session began with a practice think-aloud task, and there was a distractor task between the two main conjectures the participants received (gear and triangle). The visual field (the shapes) and scaling (placement) for the relevant and irrelevant actions was programmed using GSP, and the interview protocol was delivered using the Qualtrics survey environment. Participants completed a demographic questionnaire at the end of the session, also on Qualtrics.

Participant Demographics

Participants in both studies were undergraduate students from a large Midwestern university in the United States. Of the 47 undergraduate students in the pilot study, 17 were male and 40 were female. Participant ethnicities were Caucasian (*n* = 28), Asian (*n* = 16), and African-American (*n* = 3). The average age of participants was 19.9 years old (*sd* = 3.37), and there were 18 freshman, 16 sophomores, 7 juniors, 5 seniors, and 1 part-time student. The average GPA of participants was 3.24, and participants had an ACT-Math average of 28.1 (18 failed to respond or did not take the ACT).

Of the 57 undergraduate students who participated in the experimental study, 31 were female and 26 were male. Participant ethnicities were Caucasian (*n* = 40), Asian/Pacific Islander (*n* = 14), African American (*n* = 1), and Hispanic (*n* = 3). The average age of participants was 18.8 (*sd* = 0.87), and there were 46 freshman and 11 sophomores. Forty-four participants reported speaking English as a first language, while the remaining 13 participants reported speaking a language other than English as their first language. The average GPA of participants was 3.32, and participants had an ACT-Math average of 28.9 (15 failed to respond or did not take the ACT). In both samples, most participants had not taken any college mathematics courses past Calculus 1 and very few had taken courses beyond Calculus 2. In both samples, most participants had not taken a course covering geometry since high school

Data Sources and Analysis

All sessions were video recorded with two cameras; the pilot videos were transcribed and analyzed in the Transana analysis software. For the data from the pilot study, the type of justification each participant generated for each task (authoritative, empirical, perceptual, transformational, axiomatic – see Harel & Sowder, 1998) and its mathematical accuracy were coded. Participant's use of gesture and action while engaging in justification practices was also coded, and key excerpts were analyzed using multimodal analysis (Alibali & Nathan, in press; McNeil, 1992). Transcripts of speech and corresponding gestures for these instances were constructed based on the methods in Goodwin (2003).

For the data from the experimental study, the mathematical accuracy of each participant's justification for each task was coded. Justifications were considered accurate if they were general – i.e., they moved beyond a particular case - and if the reasoning behind the justification was sound. Using Harel and Sowder's (1998) coding scheme, transformational and axiomatic proofs were generally considered valid proofs, while authoritative, perceptual, and empirical proofs were generally considered invalid proofs. The relevant actions were designed to give insight into a transformational proof. See Appendix A for the different proof types, and the criteria used to determine whether a given justification was correct or incorrect for each task. Given that the data collection for the experimental study is ongoing, in this paper we simply present descriptive statistics of the results – bar graphs with means and error bars representing standard error of the mean. The mean values represent the percentage of correct justifications given for each condition, and performance results for the initial task and associated transfer task are collapsed (averaged together) to get overall accuracy measures for each set of tasks.

Results and Discussion

We present the results by exploring qualitative results from the pilot study and quantitative results from the experimental study for each of the four research questions.

RQ 1: How are action and gesture used spontaneously to support mathematical justification?

When asked to provide justifications for the mathematical tasks in both studies, participants used gesture and action as important tools for reasoning and communication. Harel and Sowder (1998) identify two phases of constructing a mathematical justification – the *ascertaining* phase, where the learner attempts to convince themselves about whether the conjecture is true or false, and the *persuading* phase where the learner tries to convince another that the conjecture is true or false. Gesture is certainly often a critical component of communicating an idea to others, but can spontaneous gestures support a learner in ascertaining why a mathematical property is true? In Williams et al. (under review), we discuss an instance where a participant used gesture to support their own ascertaining of a justification for the gear task – they made their hands into gears in front of them, and then reasoned through how the gears would turn by experimentally turning their hands. In other episodes in the pilot study, we saw participants spontaneously forming triangles, quadrilaterals, gears, and vertical angles using their hands and fingers, focusing on or performing actions on these embodied representations of the mathematical objects while reasoning through whether the conjecture was true or false. Other participants would describe rich mental imagery where they simulated action on the mathematical objects to arrive at their answer. Thus gesture and action seem to play an important role in the *ascertaining* phase of mathematical justification.

Figure 4 presents an excerpt from a participant justifying the inscribed quadrilateral conjecture. The participant writes a general deductive justification (line 1), and is asked for clarification by the interviewer (line 2). While responding, the participant spontaneously engages in gestures, accompanied by the rephrasing, refining, and extending of his justification (lines 3-7). The participant's gestures may be seen as overt representations of the dynamic mental imagery that underlies his understanding of the problem (Hotstetter & Alibali, 2008), indicating the importance of embodied resources in justification. The participant seemed unsure when he began talking on line 3, but by the end of his embodied explanation, he indicated he was highly confident in his answer. Here we see a participant using gesture as a tool for communication during the *persuading* phase of his mathematical justification, and using his hands to illustrate the dynamic, mental imagery that underlies his understanding of the conjecture. However, the act of engaging in this embodied justification itself seemed to strengthen his understanding of and confidence in his justification, which may suggest that performing gestures may also have helped him to think deeply about the problem in a way that went beyond his static written proof.

1 P: The arcs formed by the endpoints of the opposite angles for a circle, 360 degrees of arc. So the angles must always add up to 180 degrees because each angle has to account for half a circle- I don't know how to put this. Has to account for, uh, the re- the part of 360 degrees that the opposite angle doesn't.
((while writing verbatim))

2 I: Can you tell me what you mean by that last one?

3 P: Um, uh, if one angle -
((forms angle with right hand))

4 P: accounts for 100 degrees of arc -
((rolls left hand up from table to hold parallel to right hand))

5 P: the other one's gonna have to account for the remaining 3--uh--260 degrees of arc.
((curves left hand into arc, pulls towards angle formed by right hand))

(Slightly later in the justification)

6 P: The measure of the arc is twice - the angles have to add up to 180 –
((makes angle with two hands))

7 P: - so that the measure of the total arc, the total uh, circumference is 360.
((traces circle in air with pen 3 times))

Figure 4. Participant uses gesture to construct justification of inscribed quadrilateral conjecture

As suggested by the instance in Figure 4, gesture and action act as important media for mathematical communication. Practices of justification and argumentation are especially interesting areas of mathematics in which to study gesture, as the manner in which a justification is communicated and its convincingness to an external audience are directly relevant to determinations of its validity. Simply understanding how a conjecture works in your head is not enough to produce a valid justification – you have to be able to sensibly explain a sound and generalizable argument to others. And gestures can themselves communicate mathematically-relevant information critical to the audience’s interpretation of a justification. For example, in Williams et al. (under review) we discuss an episode where a participant is communicating a justification for the triangle conjecture by invoking a specific example of a triangle that has a side length of 5. Although the verbal component of his argument seems to be situated in an illustration of a single case – a non-generalized empirical proof scheme – his gestures give the argument generality, as he bends and stretches his hands to illustrate two sides being too short to close the triangle if the third side is longer than their sum. Thus gestures not only aid in communicating sensible and interpretable mathematical justifications – gestures can constitute key conceptual ideas in the communicative space that are not captured by speech alone.

RQ 2: Is there an implicit link between action and cognition that can support mathematical justification?

The instance in Figure 4 was from a participant who engaged in relevant actions for the inscribed quadrilateral conjecture, as shown in Table 1. The gestures the participant spontaneously used in Figure 4 when he formed the angles and traced the arcs with his hand were somewhat similar to the relevant actions performed by this participant prior to solving the problem (Table 1), although the participant reported not being consciously aware of any connection. In cases where the participant performed relevant actions and then subsequently used similar gestures when solving and communicating their justification, it is not clear whether the subsequent gestures were spontaneous, or whether they arose in part as a result of the previous directed movement. Thus the second research question is difficult to explore using qualitative data of how participants use gesture and action while engaging in justification, since the connection between the directed movement and the problem-solving is thought to be completely implicit, with its effect difficult to directly observe.

In order to address the second research question we turn to the data from the experimental study. A total of 37 participants performed relevant action for one initial task, and irrelevant action for the other initial task. The accuracy of the participants when providing justifications for the initial and transfer tasks that were associated with their actions is shown in Figure 5. All participants included in Figure 5 reported not being consciously aware of a connection between the actions they performed and the corresponding task they were asked to provide a justification for. Despite this, we see that performing relevant actions that were related to the gear task seemed to considerably boost the subsequent construction of accurate mathematical justifications.

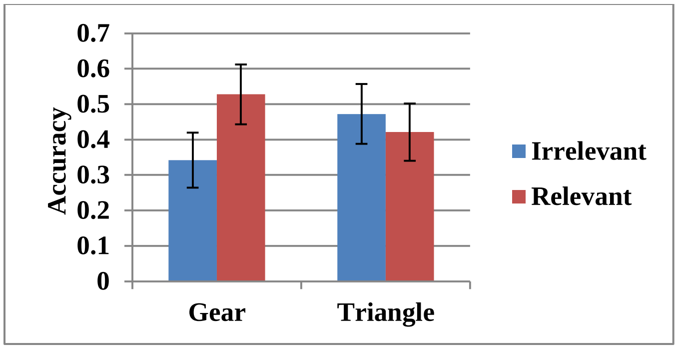


Figure 5. Average accuracy of *N* = 37 participants when justifying the gear task and gear transfer task (Gear), and triangle conjecture and triangle transfer task (Triangle) immediately after performing relevant or irrelevant action. Error bars represent standard error of the mean.

This suggests that the actions participants performed in the relevant condition of the gear conjecture (tapping back and forth) implicitly guided participants towards a valid justification. The actions participants performed for the triangle conjecture (bending and stretching arms or hands) did not seem to operate as efficiently on a purely implicit level – although as we will see later, this was mediated by viewpoint. Thus similar to Thomas and Lleras (2007; 2009), we see some evidence of an implicit link between action and cognition – that actions can produce novel insights, feeding back into the learner's cognition.

RQ 3: How can explicitly linking action-based interventions to mathematical ideas support mathematical justification?

In the mathematics classroom, connections between, for instance, an activity where students perform relevant body-based actions and a mathematical problem they attempt to understand based on those actions, are rarely left implicit. More often, the teacher will explicitly make the connection for students, or ask students to generate or figure out the connection – we previously defined this as *projection*. Here, we first investigated the impact of projection by asking each participant at the end of their session if they could generate the connection between the relevant actions they previously performed and the associated initial task. We then gave them the opportunity to provide a justification for the initial task and its transfer task a second time.

Figure 6 presents an episode in which a participant struggles to formulate a justification for the triangle conjecture when initially presented with the task. She spontaneously uses gesture to support her reasoning, but ultimately is uncertain about her answer (lines 1-4). Later (line 5), at the end of the interview session, the interviewer asks if she sees a connection between the actions she performed (Table 1) and the triangle conjecture. After thinking about it, she realizes the connection, and uses her body to form a valid embodied justification as to why the triangle conjecture is true (lines 6-10).

Figure 6 demonstrates how explicitly connecting relevant actions to mathematical ideas can be an effective and important way to ground abstract mathematical concepts in concrete body-based systems. The participant justifies the triangle theorem by labeling her left arm with the variable “A ” (line 8) and her right arm with the variable B (line 9). The *conceptual metaphor* (Lakoff & Nunez, 2000) she creates to map between her body and an imagined geometric figure allows her to formulate a generalized justification. The sensations related to the physical limitations of her body seemed to provide a resource for reasoning that simply drawing out triangles, as she had done previously, did not. In the end, the participant explicitly describes how the body-based action gave her an experiential basis from which to build her explanation (line 14).

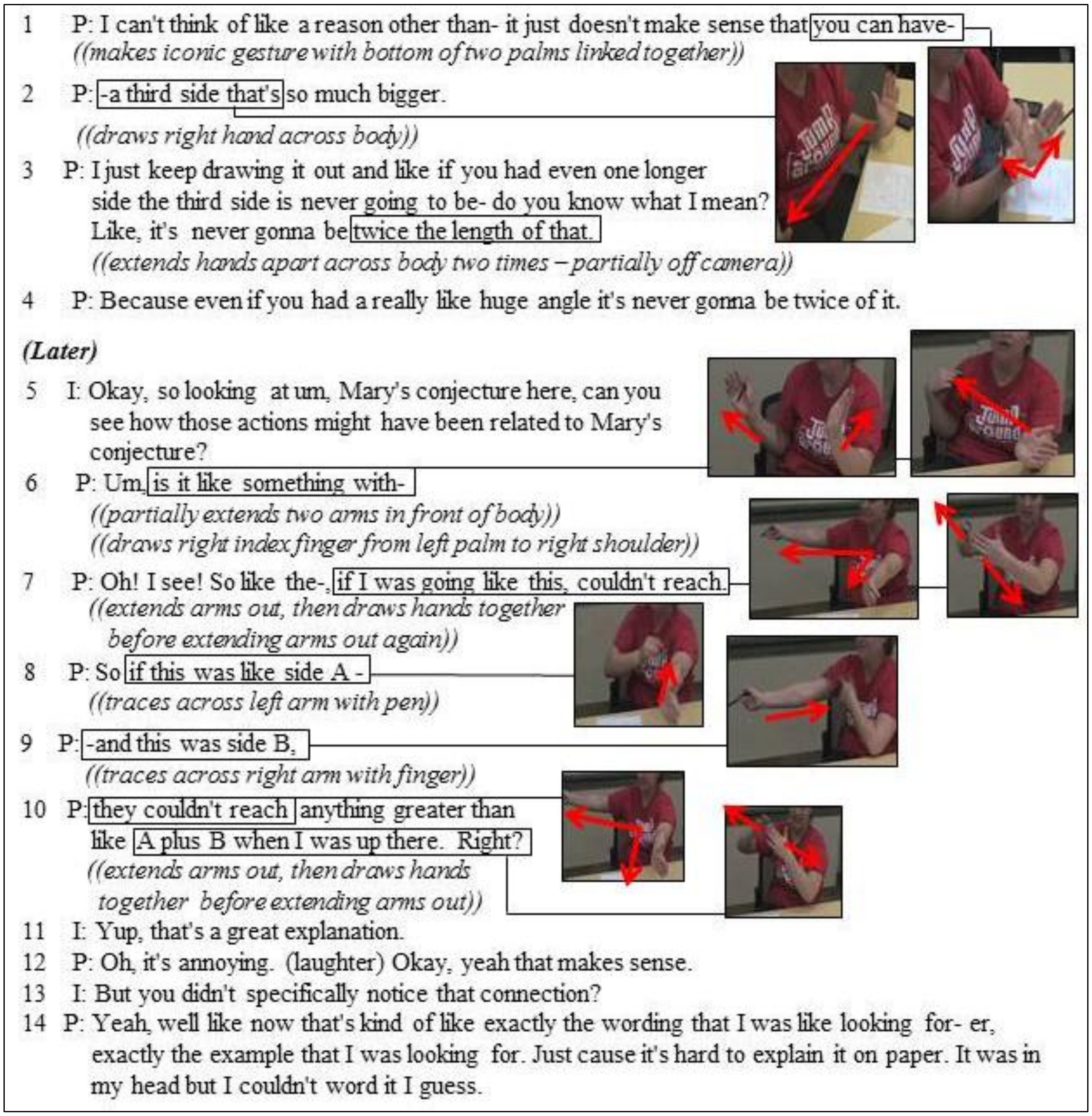


Figure 6. Participant generates an embodied proof of triangle conjecture using character viewpoint

These types of “aha” moments often occurred when participants were given the backward projection – a conjecture that they had previously struggled to justify would suddenly come into focus, and participants would imitate the relevant body-based action they had previously performed when constructing and describing their argument. The backward projection at the end of the interview seemed to be particularly effective for supporting understanding, since the participants were explicitly asked to think through and generate the connection between the actions and the tasks. However, it seemed that some participants tended to favor their initial justification made before the projection, and did not want to change an answer they had already given, despite the projection. For this reason, in the experimental study, we included a condition where the participant was given a forward projection – they were told there was a connection between the actions and the initial task *before* being given the initial task.

Results describing the impact of forward and backward projections on the construction of accurate justifications are shown in Figure 7. The bars in Figure 7 compare how performance varied when the participant performed irrelevant action and thus no projection was made (the “No Projection – Irrelevant” condition), when the participant performed relevant actions but the connection between the actions and the task was left completely implicit (the “No Projection - Relevant” condition), and when various projections were made. The “Forward Projection” condition shows accuracy on justifications when participants were attempting to provide a justification for a task for the first time, immediately after being told that there was a connection between their relevant actions and the task. The “Backward Projection” condition includes justifications given by participants who were not told there was a connection between their relevant actions and the associated task until the end of the session, and who were subsequently given a second chance to provide justifications for the initial task and transfer task based on this knowledge. The “Backward and Forward Projection” condition also includes justifications given at the end of the session when participants were given the second chance at justification, but for participants who were previously part of the “Forward Projection” condition and who already knew there was a connection. Thus these participants had received both forward and backwards projections by the end of the session.

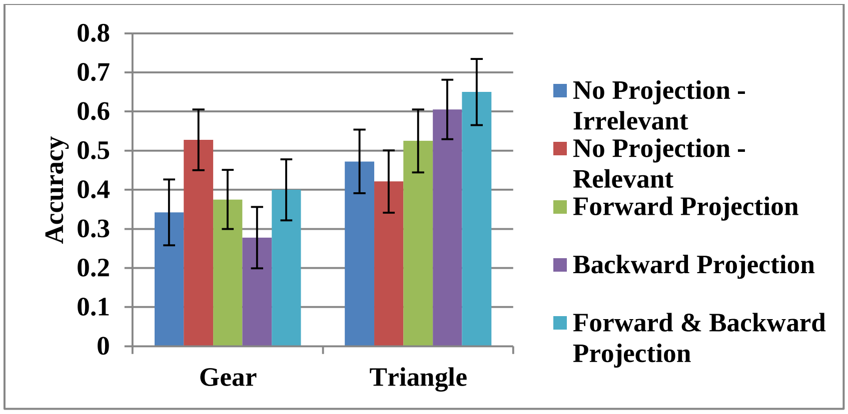


Figure 7. Average accuracy of $N = 57$ participants when justifying the gear task and gear transfer task (Gear), and triangle conjecture and triangle transfer task (Triangle) when receiving no projection, or forwards/backwards projections. Error bars represent standard error of the mean.

As can be seen from Figure 7, for the gear task, projection is not a particularly effective tool for supporting participants in building accurate justifications – in fact, we noted a number of instances of participants changing their justification from a correct justification to an incorrect justification as a result of being told about the connection. The motions of tapping back and forth are related to the task in a somewhat abstract way – embodying the idea of parity. While these simple actions may operate quite effectively on an implicit level, allowing participants to unconsciously think in terms of parity, when made explicit the connection may be too abstract or distant for the participants to grasp. For the triangle conjecture, on the other hand, the projection seemed to give the participants useful and relevant information they could use to build sound mathematical justifications, as seen in the instance in Figure 6. It is interesting that the actions that work best on an implicit level work worst on an explicit level, while the actions that are not effective when the connection is left implicit work well when the connection is made explicit through projection. Also, although the sample size is too small to draw a definite conclusion, it seems that using both forward and backwards projection is most effective in supporting students in constructing accurate justifications for the triangle conjecture – in this condition, an explicit attempt is made to maintain the cohesion of the central ideas *throughout* the interview session, using both forward and backward connections to allow participants to grasp the importance of their actions.

RQ 4: What is the role of viewpoint when using action and gesture to support practices of mathematical justification?

In the instance in Figure 6, we see an important example of how change of viewpoint can potentially support mathematical reasoning. During the participant's initial attempts to justify the conjecture (lines 1-3), we see her making observer viewpoint gestures, representing a triangle with her two palms. This reasoning is ultimately unsuccessful, and she moves on to her next problem without figuring out how to justify the triangle conjecture. Later, end the end of the interview when she is encouraged to look for a connection between her previous relevant character viewpoint actions and the conjecture, she again reasons through the problem, this time using her arms. The character viewpoint gestures she uses where she is "being the triangle" allow her to come to and communicate a valid and generalizable justification, while the previous observer viewpoint gestures were associated with an unsuccessful attempt. This pattern is typical of other participants in our sample; for example, in Srisurichan et al. (under review), we discuss another instance where a participant is able to be successful at justifying a conjecture after moving from observer to character viewpoints.

In the experimental study, directing students to perform character and observer viewpoint gestures prior to being given a conjecture was purposefully manipulated to allow for hypotheses about the strengths of different viewpoints to be tested. The results for viewpoint from the experimental study are shown in Figure 8. As can be seen from Figure 8, having participants perform character viewpoint gestures rather than observer viewpoint is associated with higher levels of subsequent performance, even when the connection between the gestures and the task is left entirely implicit. In fact, character viewpoint gestures seem to have the biggest advantage over observer viewpoint gestures when the connection is left entirely implicit (the "Relevant – No Projection" condition) and when the connection is made at the end of the interview, and participants are able to reflect on and revise their previous justification (the "Backward Projection" condition).

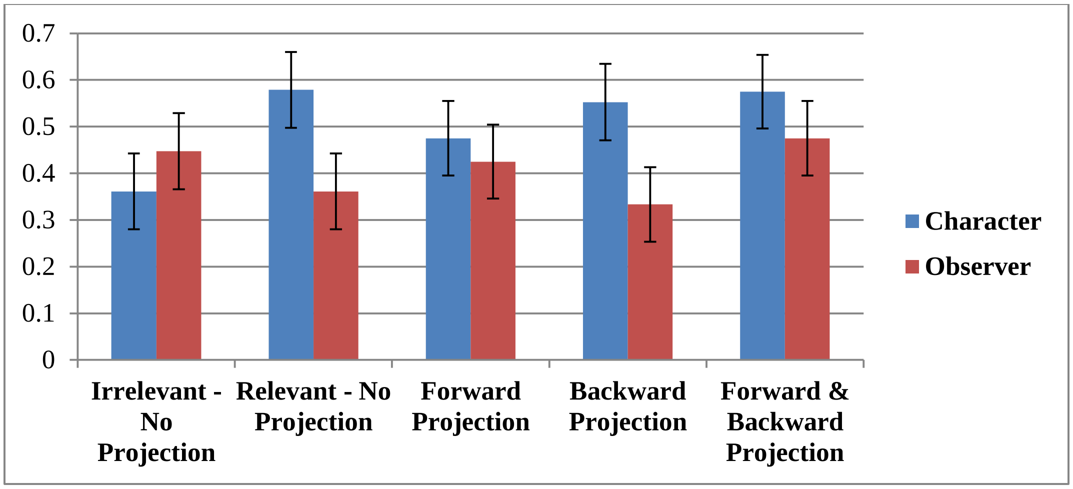


Figure 8. Average accuracy of *N* = 57 participants when providing a justification for the gear task, gear transfer task, triangle conjecture and triangle transfer task, in all projection conditions, organized by whether the participant previously performed directed character or observer viewpoint gestures. Error bars represent standard error of the mean.

Although Figure 8 seems to suggest that character viewpoint gestures are more effective across the board, there are still a number of analyses that will need to be conducted to further explore these results. First, it will be important to examine how the impact of character and observer viewpoint gestures varies with mathematical expertise – students with weaker mathematics backgrounds may benefit most from the first-person, embodied experiences of character viewpoint gestures, while students traditionally successful in mathematics class may be able to better leverage the more common observer viewpoint. Second, it will be important to separate out accuracy on the initial tasks versus the transfer tasks. Goldin-Meadow and Beilock (2010) suggest that the observer viewpoint may better promote transfer, as it strips away some of the perceptual details of the action. These will be issues we explore in future work.

Significance

Embodied action is sometimes used to ground students' understanding of elementary mathematics concepts (e.g., Abrahamson & Howsin, 2010; Goldin-Meadow et al., 2009; Martin & Schwartz, 2005); here, we provide new evidence that the body can help students understand and communicate mathematical justifications. This challenges the notion that action and gesture are simply *active* processes (Chi, 2009). Instead, we show how body-based movement can be *constructive,* allowing for the generation of new insights and novel elaborations and refinements of mathematical ideas. Body-based experience thus can provide a critical foundation from which formal and informal mathematical reasoning can be built, acting as a form of perceptual and physical grounding for challenging, abstract mathematical ideas. Further, there is evidence that action can *implicitly* direct mathematical thought, creating subconscious, embodied resources that can be brought to bear in mathematically productive ways.

There are a number of limitations to the work presented here – we only look at a few mathematical conjectures, we have only run approximately half of the participants we plan to for our experimental study, and we have not yet engaged in an inferential analysis of the experimental study results. However, the evidence we have compiled suggests that action and cognition are intimately related, and that students can learn mathematical ideas with their bodies. The implications of these results for mathematics education may be profound – if students do learn and engage in mathematical argumentation effectively with their bodies, as embodied views of the nature of cognition would suggest, an important next step will be to develop action-based interventions for classroom settings.

Action interventions where students act on the world from a character perspective, taking a close-up, first-person view of the phenomenon at hand, "feeling" the mathematics with their whole bodies, may be especially important for learning. Further, training teachers to support students in making productive and timely connections between their actions and the mathematics they are learning using projection will be a critical component of such interventions. Finally, mathematics educators should pay attention to the ways in which they and their students spontaneously use their bodies to support and communicate their mathematical thinking – these actions can instigate novel mathematical insights.

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Appendix A – Criteria for coding justifications, and determining correct/incorrect responses

Triangle Conjecture

The sum of the lengths of any two sides of a triangle must be greater than the length of the remaining side.

Say conjecture is…	Justification given	Correct/Incorrect
"False"	Any time they say it's false.	INCORRECT
"True"	**Perceptual**: Don't know why it's true; it seems like it should be true; it looks like it should usually be true	INCORRECT
"True" or "False"	**Perceptual based on Visual Field:** The justification involves the participant directly using the colored dots to construct the proof in an inappropriate way (i.e., discuss how the distance between the purple and yellow dot is 5, etc.) *Note*: This will likely only occur in projection conditions	INCORRECT
"True"	**Authoritative**: Say it's true because that's what they learned in math class; that's what their teacher/textbook says	INCORRECT
"True"	**Formulaic:** Apply Pythagorean Theorem to justify conjecture	INCORRECT
"True"	**Empirical**: Give only specific example(s), e.g. it's true for right triangles, it's true for equilateral triangles, it's true for a triangle with side lengths "1," "3," and "5"	INCORRECT
"True"	**Generalized Example (Transformational)**: Give specific example(s), but use these examples as a vehicle to show why it's true for *all* triangles. *Note*: Gestures may communicate why it's true for all triangles	CORRECT
"True"	**Action-Based Contradiction (Transformational):** Justify by saying if two sides were shorter than the remaining side, they can't reach or the triangle wouldn't be able to form/close	CORRECT
"True"	**Border Case (Transformational):** Use case where two sides are exactly equal to the third side, describing two lines lying flat on top of a third line, and the triangle not being able to open, or there being no space inside.	CORRECT
"True"	**Contextualized Contradiction (Transformational):** Justify by saying that their arms cannot reach the purple dots when the third side is too long. Justify by saying their middle fingers cannot touch when their hands are placed on the purple dots. *Note*: This will likely only occur in projection conditions	CORRECT
"True"	**Axiomatic**: Justify by saying that the shortest distance between any two points is a straight line	CORRRECT

Quadrilateral Conjecture (Transfer Task for Triangle Conjecture)

The sum of the lengths of any three sides of a quadrilateral must be greater than the length of the remaining side.

Say conjecture is…	Justification given	Correct/Incorrect
"False"	Any time they say it's false.	INCORRECT
"True"	**Perceptual**: Don't know why it's true; it seems like it should be true; it looks like it should usually be true	INCORRECT
	Authoritative: Say it's true because that's what they learned in math class; that's what their teacher/textbook says	INCORRECT
"True"	**Empirical**: Give specific example(s) only (e.g., it's true for rectangles, it's true for parallelograms)	INCORRECT
"True"	**Generalized Example (Transformational)**: Give specific example(s) (i.e., quadrilateral with sides 3, 4, 5, and 6), but use these example(s) as a vehicle to show why it's true for *all* quadrilaterals.	CORRECT
"True"	**Action-Based Contradiction (Transformational):** Justify by saying if three sides were shorter than the remaining side, they can't reach or the quadrilateral wouldn't be able to form/close	CORRECT

"True"	**Border Case (Transformational):** Use case where three sides are exactly equal to the remaining side, describing three lines lying flat on top of a fourth line, and the quadrilateral not being able to open, or there being no space inside.	CORRECT
"True"	**Contextualized Contradiction (Transformational):** Say that their arms represent 2 sides of quadrilateral and their chest is a third side. Justify by saying that their arms cannot reach the purple dots when the remaining side is too long. *Note*: This will likely only occur in projection conditions	CORRECT
"True"	**Axiomatic**: Justify by saying that the shortest distance between any two points is a straight line	CORRRECT
"True"	**Algebraic:** Justify by saying that you can divide a quadrilateral into two triangles, and then apply triangle conjecture.	CORRECT

Gear Conjecture

An unknown number of gears are connected in a chain. You know what direction the first gear turns, how could you figure out what direction the last gear turns?

Justification given	Correct/Incorrect
Perceptual: "All turn the same direction" or simply "Last gear turns in the same direction"	INCORRECT
Perceptual based on Visual Field: The justification involves the participant explaining how the diamond shapes and cross shape would behave like gears and turn together. *Note*: This will likely only occur during the final projection	INCORRECT
Perceptual with Alternation: Talks about alternation, but doesn't acknowledge that the turn direction of final gear will depend on whether the total number of gears is even or odd. They might simply say that it depends on the number of gears without mentioning even/odd. Or they might mention alternation, but say the first and last always turn the same direction. Can show alternation by: • Saying "this way" and "that way" • Pointing to left and right in alternating fashion • Making hands into gears and showing one hand turning in opposite direction of other	INCORRECT
Invalid Transformational: "Gears alternate; odd number goes in opposite direction and even number goes in same direction" or simply (i.e. gives the opposite of correct answer)	INCORRECT
"Gears alternate; need to know the number of gears to know how final gear turns, in the question says the number of gears is 'unknown.'"	INCORRECT
Transformational: "**Gears alternate**; number of gears dictates the turning direction of final gear depending on if it's **even or odd** because **odd number of gears means first and last gear go in same direction and even means first and last gear go in opposite direction**"	CORRECT (**bold = critical)**

Second Gear Conjecture (Transfer Task for Gear Conjecture)

Eleven gears are connected in a circle. Can the gears turn?

Justification given	Correct/Incorrect
Perceptual (will turn): "Chain of 11 gears will turn because all turn in same direction"	INCORRECT
Perceptual based on Visual Field: The justification involves the participant explaining how the diamond shapes and cross shape would behave like gears and turn together. *Note*: This will likely only occur during the final projection	INCORRECT
Perceptual (will turn) with Alternation: "Chain of 11 gears will turn" (but acknowledges that gears will alternate)	INCORRECT
Perceptual (won't turn): "Chain of 11 gears will not turn; I'm not sure why"	INCORRECT
Perceptual (won't turn) with Alternation: "Chain of 11 gears will not turn; I'm not sure why" (but acknowledges that gears will alternate)	INCORRECT
Transformational: "Chain of 11 gears will not turn because the gears alternate, which would mean the first and last gear go in the same direction"	CORRECT